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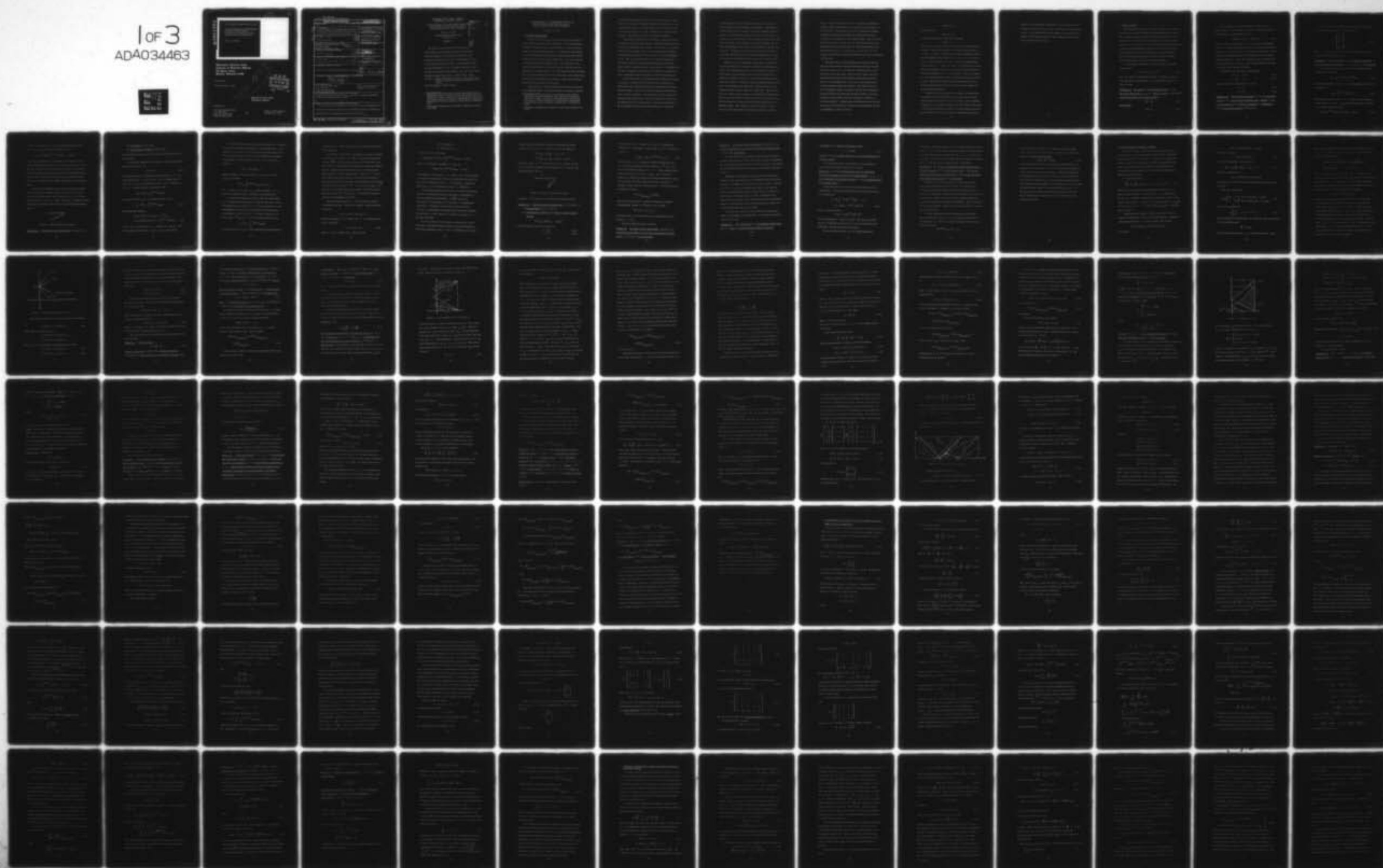

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CONTROLLABILITY AND STABILIZABILITY THEORY FOR LINEAR PARTIAL DIFFERENTIAL EQUATIONS: RECENT PROGRESS AND OPEN QUESTIONS

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CONTROLLABILITY AND STABILIZABILITY THEORY FOR LINEAR PARTIAL DIFFERENTIAL EQUATIONS: RECENT PROGRESS AND OPEN QUESTIONS †

David L. Russell‡

Technical Summary Report #1700
November 1976

ABSTRACT

This paper is an assessment of the current state of controllability and observability theories for linear partial differential equations, summarizing existing results and indicating open problems in the area. The emphasis is placed on hyperbolic and parabolic systems. Related subjects such as spectral determination, control of nonlinear equations, linear quadratic cost criteria and time optimal control are also discussed.

AMS(MOS) Subject Classification - 93BXX, 93CXX, 93DXX

Key Words - Partial differential equations, control, observation, stability, hyperbolic, parabolic, distributed control systems

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CONTROLLABILITY AND STABILIZABILITY THEORY FOR LINEAR PARTIAL DIFFERENTIAL EQUATIONS: RECENT PROGRESS AND OPEN QUESTIONS[†]

David L. Russell[‡]

1. INTRODUCTORY REMARKS

We have taken some pains in devising the title of this article to avoid committing ourselves to the impossible. It is virtually hopeless to attempt a complete outline of the development of control theory as it relates to partial differential equations. Comprehensive bibliographies in the area compiled by Johnson [44] and Robinson [85] include articles written up to 1971 and list several hundred titles. It is safe to say that a comparable number have appeared since. Such a diversity of concepts and theories leaves one in some confusion, with little idea as to how unity is to be imposed upon it. No such unification will be attempted here. We have selected one part of the whole which, with admitted bias, we feel to have some importance and which is sufficiently restricted to permit some overall pattern to be discerned.

Control theory as a whole is divided into two main sub-categories. In the AMS(MOS) Subject Classification Scheme the category of Optimal

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Control is grouped with the Calculus of Variations in Section 49 while Section 93 is devoted to "Systems, Control". Inevitably there is some overlap between these two classifications and other categories, such as Stochastic Control, e.g., could be singled out for separate listing if desired. But this basic dichotomy is generally recognized.

Historically, the first of these categories, Calculus of Variations and Optimal Control, developed first and, within the mathematical world at least, has enjoyed the wider publicity. It is, in fact, not uncommon to hear the term "optimal control" used very widely without apparent recognition that these are two words, the first very much modifying the second. This is the lingering effect of a strong "first impression" – the initial amazement of the mathematical community in seeing the subject treated seriously by no less a person than L. S. Pontrjagin [76], [77] with a subsequent revitalization of the Calculus of Variations on such a scale that the term "renaissance" is not wholly inappropriate.

In this respect there has probably been less confusion on the part of engineers, for whom the basic problem always has been that of understanding and modifying the behavior of systems whose dynamical equations include "control" terms under the influence of the plant operator – human or otherwise. From this point of view the value of optimization theory and procedure is well recognized but is not seen as the ultimate goal of the system analysis. The present article looks at the control theory of partial differential equations, or "distributed

parameter systems", from a viewpoint akin to this. We shall be concerned primarily with controllability, observability, stabilizability and general system behavior modification for certain distributed parameter systems. We will attempt to outline the theory as it has developed in the last two decades and indicate some significant problems which still remain unresolved. It is unfortunate, but almost inevitable, that the article will emphasize work either done by this author or strongly related to his work. For this one can only offer the excuse that treatment of material with which the author has not been intimately involved would of necessity lack the insight that one expects in this sort of article.

While the area of controllability, observability, stabilizability and system modification is the field of control theory traditionally important in the engineering profession, it is nevertheless true that, as a modern subject, its development can be traced in large part to the emergence of optimal control theory. For the most part (the classical Wiener filter theory is an important exception [74], [110]) optimal control theory was developed in the "state space" setting rather than in the "frequency domain" setting traditional in engineering. This meant that those who wished to take advantage of the impressive developments in optimal control were forced to translate over into the state space framework those aspects of system behavior which they already knew in the frequency domain. That "nothing was lost in translation" is still a matter of some dispute ([42]) but that much was gained can hardly be

denied. A rich structure relating control, observation, stabilization, filtering, spectral determination, and many other notions emerged in a very short time. Perhaps best known are the contributions of Kalman and Kalman and Bucy [47], [48] which have subsequently wholly revolutionized the techniques of control system design. These were ideas whose time had come. In fact, since the mathematics involved is not particularly exotic for the most part, one could argue that these were ideas which were overdue – which helps to explain their very rapid evolution once the initial steps (whose importance we would not minimize) had been taken.

The control theory of partial differential equations has followed right on the heels of that for ordinary differential equations, but with slower and heavier tread. Concerning the individual steps we shall have more to say in the sequel. Here we wish to remark, with some mystification and regret, on the historical and persistent neglect of notions "of controllability type" on the part of most mathematicians working with partial differential equations. Here again we have an area of mathematics which is somewhat "overdue" in its development. It seems possible to the author that this state of affairs is in part attributable to the persistent influence of Hadamard's definition of a "well-posed problem". (Perhaps with some historians we can see that every great act of creation has a tendency to foster limitations as well as possibilities.) Given an equation

$$L(u, f, x, t) = 0 \ ,$$

initial conditions

$$J(u(\cdot, 0), g) = 0$$

and boundary conditions (or other side conditions)

$$B(u, h) = 0 \ .$$

with f, g and h known functions, well-posedness requires that the solution $u = w(x, t)$ should behave in a suitably "regular" manner as f, g and h are varied. With this point of view the emphasis is placed upon establishing that the influence of f, g and h on u is "not too great" – modest changes in f, g and h producing correspondingly modest changes in u. When f, g and/or h are regarded as controls, however, the emphasis shifts and we want to know that the influence of the control functions on u is "not too little" – that a given change in w can indeed be realized with an appropriate change in the control function. Thus we have diametrically opposite emphasis in control theory from that which has historically motivated those working with partial differential equations.

It will be seen in the sequel that our subject, particularly from the point of view of observation theory, has some points in common with the relatively new field of ill-posed problems. This is explicitly recognized, e.g., by Seidman in [102]. The recent interest in this field together with the developments which we describe represent at least in part "the other side of the coin" relative to the historically dominant

emphasis on well-posedness and regularity. One may hope for the development of general recognition of the complementarity of the two points of view. To the degree that this article assists in encouraging such an attitude and in stimulating interest in these matters generally we will count our present effort a success.

2. BASIC CONCEPTS

While most of this paper will deal with "concrete" control systems described by comparatively simple linear partial differential equations, it is nevertheless useful to introduce some fundamental ideas in a more general setting. We will begin by reviewing, very briefly, the fundamental notions of control theory as they apply to finite dimensional linear constant coefficient systems and will then indicate how these notions may be extended to systems in Hilbert spaces. (Most ideas extend also to Banach spaces [15], [16] but we restrict attention to Hilbert spaces in this paper to avoid less essential difficulties.)

We begin with consideration of the linear constant coefficient control system

$$\dot{x} = Ax + Bu, \quad x \in E^n, \quad u \in E^m \tag{2.1}$$

with A, B matrices of appropriate dimension. We will be concerned with application of control functions $u \in L^2([0,T]; E^m)$ where T is a positive number.

<u>Definition 2.1.</u> <u>The system</u> (2.1) <u>is controllable (in time</u> $T > 0$) <u>if and only if, given points</u> $x_0, x_1 \in E^n$, <u>there is a control</u> $u \in L^2([0,T]; E^m)$ <u>such that the solution of (2.1) which satisfies</u>

$$x(0) = x_0 \tag{2.2}$$

<u>also satisfies</u>

$$x(T) = x_1 \; . \tag{2.3}$$

It is convenient to use the terminology "u steers x_0 to x_1 (during $[0,T]$)". It is well known [K] (and we do not repeat the proof here) that (2.1) is controllable just in case

$$\operatorname{rank}[B, AB, A^2B, \ldots, A^{n-1}B] = n . \tag{2.4}$$

As this condition is independent of T we can see that controllability of (2.1) is a property which either holds or does not hold for all intervals $[0,T]$, $T > 0$, and we may speak of (2.1) as being "controllable" without reference to any particular interval. This property does not generally extend to systems $\dot{x} = A(t)x + B(t)u$ with time varying coefficients and, as we shall see does not extend to certain partial differential equations even if we do have constant coefficients.

Paired with (2.1) is the linear observed system

$$\dot{y} = Cy, \qquad y \in E^n , \tag{2.5}$$

$$\omega = Hy, \qquad \omega \in E^r , \tag{2.6}$$

which is "dual" to (2.1) just in case $r = m$ and

$$C = -A^* \tag{2.7}$$

$$H = B^* . \tag{2.8}$$

Definition 2.2. The linear observed system (2.5), (2.6) is observable (in time $T > 0$) just in case the following is true: whenever y is a non-zero solution of (2.5) then the "observation" ω, determined by (2.6), is not the zero element of $L^2([0,T]; E^r)$.

One can show rather easily [55] that (2. 5), (2. 6) is observable (on an arbitrary interval $[0, T]$, $T > 0$) just in case

$$\operatorname{rank} \begin{bmatrix} H \\ HC \\ HC^2 \\ \vdots \\ HC^{n-1} \end{bmatrix} = n . \tag{2.9}$$

When $r = m$ and (2. 7), (2. 8) are true the rank conditions (2. 4), (2. 9) are evidently equivalent and we have

Proposition 2. 3. The control system (2. 1) is controllable if and only if the dual linear observed system (2. 5), (2. 6), (2. 7), (2. 8) is observable.

It is known ([47], [60], [103], [105]) that controllability of (2. 1) is equivalent to the matrix

$$Z(T) = \int_0^T e^{A(T-t)} BB^* e^{A^*(T-t)} dt \tag{2.10}$$

being positive definite for every $T > 0$ and observability of (2. 5), (2. 6) is equivalent to

$$W(T) = \int_0^T e^{C^* t} H^* H e^{Ct} dt \tag{2.11}$$

being positive definite for every $T > 0$. If $Z(T)$, given by (2. 10), is positive definite, a control u satisfying the requirements of Definition 2. 1 can be given explicitly ([G]) by

$$u(t) = B^* e^{A^*(T-t)} Z(T)^{-1}(x_1 - e^{AT} x_0), \quad t \in [0, T] . \tag{2.12}$$

If $W(T)$, given by (2.11), is positive definite the solution $y(t)$ of (2.5) can be recovered from the observation $\omega(t)$ via

$$y(t) = e^{Ct} W(T)^{-1} \int_0^T e^{C^* s} H^* \omega(s)ds, \qquad t \in [0, T] \ .$$

A very elegant formulation of controllability, observability, and the duality relationship between the two, has been set forth by S. Dolecki [15], [16]. The basic idea was presented by Dolecki in an unpublished report and was later incorporated with work of the present author in the joint paper [16]. While [15], [16] treat the general problem in Banach space we here consider only the Hilbert space theory for reasons noted above.

We let X, Y, Z be Hilbert spaces (over the same field, ordinarily the real numbers or the complex numbers) and we let $S: X \to Z$ be a bounded linear operator and $C: \mathcal{D}(C)$ (domain of C) $\subseteq Y \to Z$ a linear operator with $\mathcal{D}(C)$ dense in Y. Then $\{X, Y, Z, S, C\}$ together constitute an abstract linear control system which we set forth diagrammatically in Fig. 2.1

$$\begin{tikzcd} Y \supseteq \mathcal{D}(C) \arrow[rr, "C"] & & Z \\ X \arrow[urr, "S"] & & \end{tikzcd}$$

Figure 2.1. Abstract linear control system

<u>Definition 2.4.</u> <u>The abstract linear control system</u> $\{X, Y, Z, S, C\}$ <u>is</u>

(i) <u>controllable if</u> $\mathcal{R}(C) \supseteq \mathcal{R}(S)$;

(ii) <u>approximately controllable if</u> $\overline{\mathcal{R}(C)} \supseteq \mathcal{R}(S)$.

Some intermediate notions can also be introduced but these will suffice for the moment.

To indicate the application of this theory in a setting slightly more general than (2.1), let A in

$$\dot{z} = Az + Bu \tag{2.13}$$

now be the generator of a semigroup of bounded operators, e^{At}, $t \geq 0$, on the Hilbert space Z. Let the control u lie in a Hilbert space U and let $B: U \to Z$ be a bounded linear operator. Let $Y = L^2([0,T]; U)$, $X = Z \oplus Z$. From the variation of parameters formula

$$z(t) = e^{At} z(0) + \int_0^t e^{A(t-s)} Bu(s)ds$$

we see that u steers z_0 to z_1 during $[0,T]$ just in case

$$z_1 - e^{AT} z_0 = \int_0^T e^{A(T-s)} Bu(s)ds \ . \tag{2.14}$$

This prompts the definitions:

$$S: X(= Z \oplus Z) \to Z: \ S(z_0, z_1) = z_1 - e^{AT} z_0 \ ;$$

$$C: Y(= L^2([0,T]; U)) \to Z: \ Cu = \int_0^T e^{A(T-s)} Bu(s)ds \ . \tag{2.15}$$

If (i) of Def. 2.4 holds, there is a u for which $Cu = S(z_0, z_1)$. Then (2.14) obtains and the control u steers z_0 to z_1 during $[0,T]$.

In the case of finite dimensional systems (2.1) with S, C constructed just as above, (i) and (ii) of Definition 2.4 are equivalent since a dense subspace of E^n must be E^n itself. But in infinite dimensional spaces there is a genuine distinction between controllability and approximate controllability. If (ii) is true, then for every $\varepsilon > 0$ we can find u_ε such that

$$\| Cu_2 - S(z_0, z_1) \|_Z < \varepsilon \ ,$$

which, returning to (2.14), means there is a vector $z_\varepsilon \in Z$ with $\| z_\varepsilon \|_Z < \varepsilon$ such that

$$e^{AT} z_0 + \int_0^T e^{A(T-s)} Bu_\varepsilon(s)ds = z_1 + z_\varepsilon \ ,$$

i.e. u_ε steers z_0 to a point $z_1 + z_\varepsilon$ within a distance ε of z_1. This is the accepted meaning of approximate controllability [21].

It is clear from Definition 2.4 that we obtain the same property if S is replaced by another operator S_1 with the same range. Thus in the example above we could take $X = Z$ and S_1 the identity on Z and controllability of $\{Z, Y, Z, S_1, C\}$ would amount to the same thing as controllability of $\{Z \oplus Z, Y, Z, S, C\}$. However, if we take $X = Z$ and let $S_1 z_0 = -e^{AT} z_0$, $z_0 \in Z$, then $\mathcal{R}(C) \supseteq \mathcal{R}(S_1)$ means only that

$$0 = e^{AT} z_0 + \int_0^T e^{A(T-s)} Bu(s)ds$$

can be solved for each $z_0 \in Z$, which means that each initial state can

be steered to zero. This is not the same as our earlier property unless e^{AT} is invertible.

The meaning of approximate controllability also varies, depending on our choice of the operator S_1. With S_1 equal to the identity the condition $\overline{\mathcal{R}(C)} \supseteq \mathcal{R}(S_1)$ means the system can be steered from a zero initial state to an arbitrarily small neighborhood of any desired terminal state (or, equivalently from any initial state to an arbitrarily small neighborhood of any desired terminal state). With $S_1 = -e^{AT}$ the condition $\overline{\mathcal{R}(C)} \supseteq \mathcal{R}(S_1)$ means that the system can be steered from an arbitrary initial state to an arbitrarily small neighborhood of any terminal state which lies in the range of e^{AT}. In most cases the range of e^{AT} is dense in Z and the two mean the same thing. But there are cases where the range of e^{AT} is not dense in Z and then the two types of approximate controllability are different.

By using various operators S, C one can discuss a variety of controllability concepts. Consider, for example, a slight modification of the above

$$\dot{z} = Az + Bu + Dv, \quad z(0) = z_0 \in z \;,$$

with the "disturbance" v in a Hilbert space V. We suppose that we have an observation

$$\omega = Hz, \quad H: Z \to W \tag{2.16}$$

where W is also a Hilbert space. This time we let

$$X = Z \oplus L^2([0,T]; V)$$

$$Z_\varepsilon = L^2([\varepsilon, T]; W), \quad \varepsilon > 0 \ ,$$

and define $S: X \to Z_\varepsilon$ by setting

$$(S(z_0, v))(t) = H e^{At} z_0 + H \int_0^t e^{A(t-s)} D v(s) ds, \quad t \in [\varepsilon, T] \ .$$

We let $Y = L^2([0,T]; U)$ and define $C: Y \to Z_\varepsilon$ by

$$(Cu)(t) = H \int_0^t e^{A(t-s)} Bu(s) ds, \quad t \in [\varepsilon, T] \ .$$

Controllability of $\{X, Y, Z_\varepsilon, S, C\}$, i.e. $\mathfrak{R}(C) \supseteq \mathfrak{R}(S)$, now corresponds to the property of being able to reduce the observation (2.16) to zero on $[\varepsilon, T]$ with an appropriate control $u \in Y = L^2([0,T]; U)$, whatever the initial state $z_0 \in Z$ and disturbance function $v \in L^2([0,T]; V)$.

One point which should be made is this. As with meat and poison, one man's controllability (sometimes called exact controllability) is another's approximate controllability. If $\overline{\mathfrak{R}(S_1)} \supseteq \mathfrak{R}(S)$ and $\{X, Y, Z, S_1, C\}$ is controllable, then $\{X, Y, Z, S, C\}$ is approximately controllable. It frequently occurs, however, that we can show $\{X, Y, Z, S, C\}$ approximately controllable without being able to identify any reasonably "concrete" operator S_1 for which $\{X, Y, Z, S_1, C\}$ is (exactly) controllable.

It is fairly well known that control and observation are "dual to each other". The abstract framework permits a rigorous presentation of this duality relationship. Since S and C of Definition 2.4 have been

assumed bounded and with dense domain, respectively, the adjoint operators $S^*: Z \to X$ and $C^* = \mathcal{D}(C^*) \subseteq Z \to Y$ may be defined by

$$(S^* z, x)_X = (z, Sx)_Z, \quad x \in X, \quad z \in Z \ ,$$

$$(C^* z, y)_Y = (z, Cy)_Z, \quad y \in \mathcal{D}(C), \quad z \in \mathcal{D}(C^*) \ ,$$

the domain $\mathcal{D}(C^*)$, of C^* being the set of $z \in Z$ for which $(z, Cy)_Z$ can be extended to a bounded linear functional on Y. We present this diagrammatically in Fig. 2.2

$$\begin{tikzcd} Y & \mathcal{D}(C^*) \subseteq Z \arrow[l, "C^*"'] \arrow[dl, "S^*"] \\ X & \end{tikzcd}$$

Figure 2.2. Abstract linear observed system

and refer to $\{C^*, S^*, Z, Y, X\}$ as an abstract linear observed system.

Definition 2.5. The abstract linear observed system $\{C^*, S^*, Z, Y, X\}$ is

(i) distinguishable if $\ker C^* \subseteq \ker S^*$;

(ii) observable if, in addition to (i), there is a positive number K such that

$$\|C^* z\|_Y \geq K \|S^* z\|_X, \quad z \in \mathcal{D}(C^*) \ .$$

The linear observed system dual to (2.13) is

$$\dot{\zeta} = -A^* \zeta \tag{2.17}$$

$$\omega = B^* \zeta \ . \tag{2.18}$$

If we choose $X = Z$, S = identity on Z, then S^* is likewise the identity on Z. If we define C by (2.15) then $C^*: Z \rightarrow Y = L^2([0,T]; U)$ is given by

$$(C^* z_1)(t) = \omega(t) = B^* e^{A^*(T-t)} z_1, \quad z_1 \in Z. \tag{2.19}$$

We may regard $\zeta_1 \in Z$ as supplying a terminal condition for (2.17) ($-A^*$ generates a semigroup of bounded operators e^{-A^*t} for $t \leq 0$), the terminal condition being given at $t = T$. In this setting (i) above merely says that if $\zeta_1 \neq 0$ then $(C^* \zeta_1)(t) = \omega(t)$ is not zero in $L^2([0,T]; U)$, which in turn implies that different terminal states $\zeta_1, \hat{\zeta}_1$ for (2.17) yield, via (2.18), different observations $\omega(t)$, $\hat{\omega}(t)$, respectively. Hence the term "distinguishability". On the other hand (ii) goes further to say that

$$\|C^* \zeta_1\|_{L^2([0,T]; U)} \geq K \|\zeta_1\|_Z .$$

This is much stronger since it implies the existence of a bounded "reconstruction" operator $R = L^2([0,T]; U) \rightarrow Z$ for which

$$R\omega = RC^* \zeta_1 = \zeta_1, \quad \zeta_1 \in Z ,$$

so that the state ζ_1 can be continuously reconstructed from the observation $\omega = C^* \zeta_2$.

In [16] the following theorems are proved.

Theorem 2.6. *The abstract linear control system* $\{X, Y, Z, S, C\}$ *is approximately controllable if and only if the abstract linear observed system* $\{C^*, S^*, Z, Y, X\}$ *is distinguishable.*

Theorem 2.7. The abstract linear control system $\{X, Y, Z, S, C\}$ is (exactly) controllable if and only if the abstract linear observed system $\{C^*, S^*, Z, Y, X\}$ is observable.

We remark that most controllability results for linear systems are proved by a method which, in direct application, amounts to establishing the corresponding observability property for the dual system. This pattern will be observed repeatedly in the sequel but some exceptions will also be noted.

Perhaps the most important "message" of the foregoing abstract treatment is that controllability and observability questions are familiar questions of solvability of linear equations in function space. Theorems 2.6 and 2.7, in fact, are just paraphrased versions of the standard functional analysis theorems which relate the range of an operator to the null space of its adjoint (see, e.g. [17]). Nevertheless, the questions we ask, and the case we make of the answers are sufficiently distinct and particular that we can legitimately regard our subject as a separate mathematical area in its own right.

One of the main concerns of control theory is the relationship between controllability and stabilizability. For the finite dimensional system (2.1) we have

Definition 2.8. The control system (2.1) is stabilizable if there exists an $m \times n$ matrix K such that the linear feedback control law

$$u = Kx \quad ,$$

substituted in (2.1), yields a "closed loop" system

$$\dot{x} = (A + BK)x \tag{2.20}$$

in which $A + BK$ is a stability matrix, i.e., all of its eigenvalues have negative real parts.

The relationship between controllability and observability for finite dimensional systems is well known. We have

Theorem 2.9. If (2.1) is controllable then it is also stabilizable; moreover, it is possible to prescribe any complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ as the eigenvalues of the closed loop matrix $A + BK$ by appropriate choice of the feedback matrix K.

Perhaps the nicest proof of the first part of the theorem is due, independently, to Lukes [60] and Kleinman [49] who showed that if (2.1) is controllable then

$$K = -B^*\left(\int_0^T e^{-At} BB^* e^{-A^*t} dt\right)^{-1}, \qquad T > 0 ,$$

$$= (\text{cf. } (2.10)) = -B^* e^{A^*T} Z(T)^{-1} e^{AT} \tag{2.21}$$

yields a closed loop matrix

$$A + BK = A - BB^* e^{A^*T} Z(T)^{-1} e^{AT}$$

whose eigenvalues have negative real parts. The second part of the theorem is established by reducing (2.1) to control canonical form (see [55], [111]). We shall carry this out in Section 4.

It is not in general true that if (2.1) is stabilizable then it is

controllable. One need only take A to be a stability matrix at the outset and $B = 0$ for a counter-example. However, the "pole placement property", i.e., being able to place the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the closed loop matrix $A + BK$ at will in the complex plane does imply controllability. In fact, a somewhat weaker condition implies controllability:

Theorem 2.10. *If (2.1) can be stabilized both forward and backward, i.e. if feedback matrices $\tilde{K}$, $\hat{K}$ can be found such that $A + B\tilde{K}$ has only eigenvalues with negative real parts while $A + B\hat{K}$ has only eigenvalues with positive real parts, then (2.1) is controllable.*

Theorem 2.10 is not very significant in connection with finite dimensional systems since even more is true. If no eigenvalue of $A + BK$ is invariant with respect to all admissible feedback matrices K then (2.1) is controllable. But Theorem 2.10 is important in the sense that its extension to infinite dimensional systems sometimes provides the only means available for establishing controllability. This is particularly true for certain hyperbolic systems to be discussed in Section 5. Theorem 2.10 is proved in [93].

For finite dimensional systems the property that all solutions $x(t)$ of (2.20) tend to zero as $t \to +\infty$ is equivalent to the property of all eigenvalues of $A + BK$ having negative real parts, which in turn can be used to show that

$$\|e^{(A+BK)t}\| \leq Me^{-\gamma t}, \quad t \geq 0$$

for certain positive constants M, γ, depending only on the matrix $A + BK$. Thus, under such circumstances, solutions of (2.20) have the property of <u>uniform exponential decay</u>:

$$\|x(t)\| \leq Me^{-\gamma t} \|x(0)\| \quad . \tag{2.22}$$

For infinite dimensional systems the situation is not as simple. It is possible to have all solutions of the system tending to zero without any uniform decay rate such as (2.22) applying. Perhaps not surprizingly, controllability in the sense of (i) of Definition 2.4 can be associated with uniform exponential decay in certain distributed parameter systems while approximate controllability, (ii) of Definition 2.4, can be associated only with the weaker notion of all solutions tending to zero as $t \to +\infty$. The results thus far obtained along these lines pertain, for the most part, to specific types of systems and are left for treatment in later sections of this paper.

3. LINEAR SYMMETRIC HYPERBOLIC SYSTEMS

Perhaps the simplest of distributed parameter systems are those described by linear symmetric hyperbolic systems of first order partial differential equations in two variables t (time) and x (a single space variable). They have the additional advantage, from the didactic viewpoint, of possessing sufficient complexity to permit exposition of a wide variety of questions which are of interest to those working with distributed parameter control.

We consider the system

$$\frac{\partial \tilde{w}}{\partial t} = \tilde{A}(x) \frac{\partial \tilde{w}}{\partial x} + \tilde{B}(x)w, \quad w \in E^n, \quad t \geq 0, \quad 0 < x < 1 \ , \tag{3.1}$$

wherein $\tilde{A}(x)$ is a continuously differentiable real symmetric $n \times n$ matrix function and $\tilde{B}(x)$ is a real continuous $n \times n$ *matrix function* for $0 \leq x \leq 1$. We further suppose that the eigenvalues $\lambda_k(x)$ of $\tilde{A}(x)$ satisfy $\lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_p(x) < 0 < \lambda_{p+1}(x) \leq \lambda_{p+2}(x) \leq \cdots \leq \lambda_{p+q}(x)$, $0 \leq x \leq 1$ $(p+q = n)$; and: $\{\exists\, x \in [0,1] \ni \lambda_k(x) = \lambda_{k+1}(x)\} \Rightarrow \{\forall x \in [0,1],\ \lambda_k(x) = \lambda_{k+1}(x)\}$.

A lemma proved by Phillips [75] allows us to reduce (3.1) to a convenient standard form. There it is shown that there is a continuously differentiable orthogonal $n \times n$ matrix function $O(x)$ such that

$$O(x)^{-1}\, \tilde{A}(x)\, O(x) = O(x)^T\, \tilde{A}(x)\, O(x) \equiv A(x)$$

is diagonal:

$$A(x) = \mathrm{diag}(\lambda_1(x),\ \lambda_2(x),\ \ldots,\ \lambda_n(x)) .$$

The change of variable

$$\tilde{w}(x,t) = O(x)\ w(x,t) \tag{3.2}$$

in (3.1) gives the equation in standard form

$$\frac{\partial w}{\partial t} = A(x)\frac{\partial w}{\partial x} + B(x)w, \quad t \geq 0, \quad 0 < x < 1 , \tag{3.3}$$

with $A(x)$ as indicated and

$$B(x) = O(x)^T[\tilde{B}(x)\ O(x) + \tilde{A}(x)\ O'(x)] .$$

We note that $A(x)$ is still continuously differentiable and $B(x)$ is still continuous.

Now $A(x)$ can be written

$$A(x) = \begin{pmatrix} A^-(x) & O \\ O & A^+(x) \end{pmatrix}, \quad \begin{aligned} A^-(x) &= \mathrm{diag}(\lambda_1(x),\ldots,\lambda_p(x)) , \\ A^+(x) &= \mathrm{diag}(\lambda_{p+1}(x),\ldots,\lambda_{p+q}(x)) , \end{aligned} \tag{3.4}$$

and w can be partitioned as

$$w = \begin{pmatrix} w^- \\ w^+ \end{pmatrix}, \quad w^- \in E^p, \quad w^+ \in E^q . \tag{3.5}$$

The "characteristics" corresponding to the eigenvalue $\lambda_k(x)$ are solutions of the differential equation

$$\frac{dx}{dt} = -\lambda_k(x) .$$

These are denoted generally by c_k, or more specifically by $c_k(x,t)$

if we wish to specify the characteristic which passes through the point (x, t). On c_k the k-th component, w^k, of w satisfies the linear ordinary differential equation

$$\frac{d}{dt}[w^k(x(t), t)] \equiv \frac{\partial w^k}{\partial t}(x(t), t) - \lambda_k(x(t)) \frac{\partial w^k}{\partial x}(x(t), t)$$

$$= B^k(x(t))\, w(x(t), t), \qquad k = 1, 2, \ldots, n\ , \tag{3.6}$$

where B^k denotes the k-th row of B. These n equations are, in general, coupled by the terms $B^k w$ and, since they are valid on different characteristics c_k, this coupling is of a more complex variety than that normally encountered in systems of ordinary differential equations. Initial values at $t = 0$ are provided when one specifies

$$w(\cdot, 0) = w_0 \in (L^2[0, 1]; E^n)\ . \tag{3.7}$$

Consider now a point $(0, t_0)$ as shown in Fig. 3.1. At $(0, t_0)$ the "incoming information" consists of values of w^k associated with characteristics $c_k(0, t_0)$, $k = p+1, \ldots, p+q = n$ which reach $(0, t_0)$ after passing through the region $t < t_0$, $x > 0$. On the other hand, the "outgoing information" consists of values of w^k associated with characteristics $c_k(0, t_0)$, $k = 1, 2, \ldots, p$, leaving $(0, t_0)$ to enter the region $t > t_0$, $x > 0$. Referring back to (3.4), (3.5) we see that, along the boundary $x = 0$, w^+ is the incoming information and w^- is the outgoing information. It is easy to see that, along the boundary $x = 1$, the roles of w^- and w^+ are just reversed.

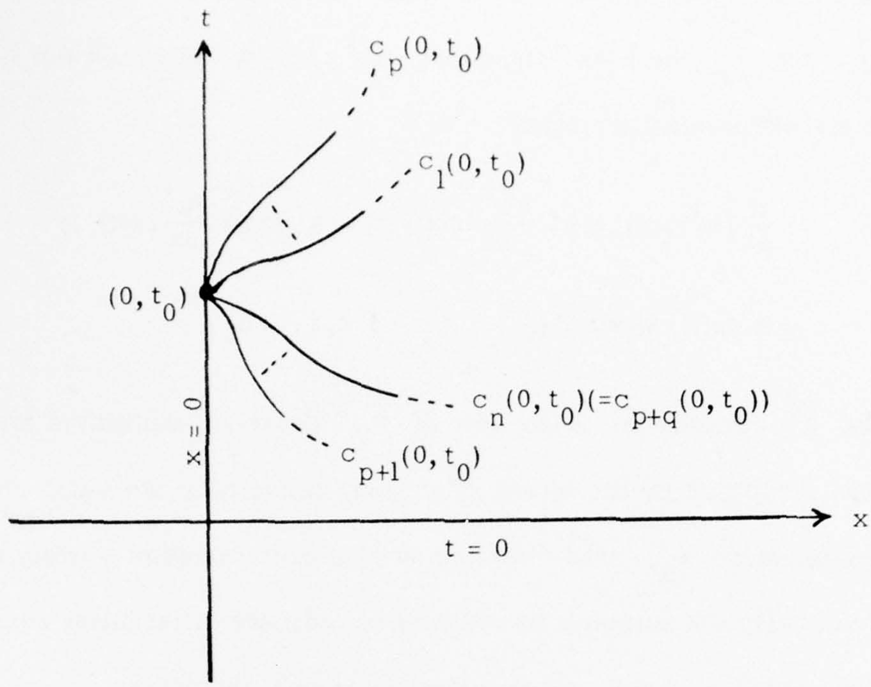

Figure 3.1. Boundary configuration at $x = 0$

If the boundary conditions for the original system (3.1) are written as

$$\tilde{C}_0 \, \tilde{w}(0,t) = 0, \quad \tilde{C}_1 \, \tilde{w}(1,t) = 0 , \tag{3.8}$$

then, under (3.2), they transform to

$$\tilde{C}_0 \; O(0) \; w(0,t) \equiv C_0 \; w(0,t) = 0 ,$$

$$\tilde{C}_1 \; O(1) \; w(1,t) \equiv C_1 \; w(1,t) = 0 ,$$

and C_0, C_1 can be partitioned in agreement with (3.5) to give

$$C_{0-} \; w^-(0,t) + C_{0+} \; w^+(0,t) = 0 , \tag{3.9}$$

$$C_{1-} \; w^-(1,t) + C_{1+} \; w^+(1,t) = 0 . \tag{3.10}$$

The basic requirement for existence and uniqueness of solutions of (3.1), (3.8) (or (3.3), (3.9), (3.10)) is that at each boundary point the outgoing information should be determined by the incoming information. This implies that (3.9) can be solved for $w^-(0,t)$ and (3.10) can be solved for $w^+(1,t)$:

$$w^-(0,t) = D_0\, w^+(0,t) \quad , \tag{3.11}$$

$$w^+(1,t) = D_1\, w^-(1,t) \quad . \tag{3.12}$$

The type of control which we will consider in this section is boundary value control. When control is exercised the second equation in (3.8) takes the form

$$\tilde{C}_1\, \tilde{w}(1,t) + \tilde{D}\, u(t) = 0, \qquad u \in E^q \quad ,$$

and we suppose that, after the transformation indicated above, this results in (3.12) being replaced by

$$w^+(1,t) = D_1\, w^-(1,t) + D\, u(t) \tag{3.13}$$

with D a nonsingular $q \times q$ matrix. This means that the control vector u "dominates" the outgoing information at $x = 1$.

The basic existence and uniqueness theorem for solutions is (see [11], [17], [31]).

Theorem 3.1. The linear operator

$$Lw = A \frac{\partial w}{\partial x} + Bw \tag{3.14}$$

defined on the domain in $L^2([0,1]; E^n)$ consisting of functions $w \in H^1([0,1]; E^n)$ which satisfy boundary conditions of the form (3.11),

(3.12) generates a semigroup $S(t)$ of bounded operators on $L^2([0,1]; E^n)$ (group if D_0 and D_1 are both invertible, which can be true only if $p = q = n/2$). Given an initial state (3.7), the system (3.3), (3.11), (3.12) has the generalized solution $w \in \mathbf{C}([0,\infty); L^2([0,1]; E^n))$ given by

$$w(\cdot, t) = S(t)\, w_0 \quad .$$

When the boundary condition (3.12) is replaced by (3.13) we again obtain a generalized solution $w \in \mathbf{C}([0,\infty); L^2([0,1]; E^n))$ of the general form

$$w(\cdot, t) = S(t) w_0 + C(t)\, u^{(t)} \tag{3.15}$$

where $u^{(t)}$ denotes the restriction of u to $[0,t]$ and $C(t)\colon L^2([0,t]; E^q) \to L^2([0,1]; E^n)$ is a bounded linear operator for each t.

We remark that smoother solutions can be obtained if w, u satisfy certain consistency conditions and w_0 is smooth. See [11], [17], [31] for details. It is known from general semigroup theory that

$$\|S(t)\| \leq M\, e^{\gamma t}, \qquad t \geq 0 \ ,$$

for some M, γ depending only on $A(x)$, $B(x)$, D_0, D_1. A specific estimate of M, γ appears in [91]. Then (3.15) gives

$$\|w(\cdot, t)\|_{L^2([0,1];E^n)} \leq M\, e^{\gamma t} \|w_0\|_{L^2([0,1];E^n)} + \rho_t \|u\|_{L^2([0,t];E^q)} \tag{3.16}$$

We are now in a position to state the control problem for the control system (3.3), (3.11), (3.13):

Control Problem. Given $w_0, w_1 \in L^2([0,1]; E^n)$ and $T > 0$, to determine, if possible, $u \in L^2([0,T]; E^q)$ such that the solution w of (3.3), (3.7), (3.11), (3.13) also satisfies

$$w(\cdot, T) = w_1 . \tag{3.17}$$

(i.e. to "steer" solution of (3.3), (3.11), (3.13) from w_0 to w_1 during $[0,T]$).

It turns out that this problem can always be solved if T is sufficiently large in the case where $S(t)$ described in Theorem 3.1, is a group. In other cases the results are more complex and w_1 must frequently be further restricted.

We will begin with a brief description of a constructive method whereby certain controllability results can be obtained. This will be followed by other theorems which are obtained by more indirect procedures.

Theorem 3.2. Let

$$T \geq \int_0^1 \frac{dx}{\lambda_{p+1}(x)} - \int_0^1 \frac{dx}{\lambda_p(x)} . \tag{3.18}$$

Then the above control problem can be solved for the case $w_1 = 0$. If $S(t)$ is a group, i.e., $p = q = n/2$ and D_0, D_1 are invertible, then the control problem can be solved for general $w_1 \in L^2([0,1]; E^n)$.

Sketch of Proof. The significance of the inequality (3.18) lies in the fact that when it is satisfied the rectangular region $0 \leq x \leq 1$, $0 \leq t \leq T$ can be divided into three separate regions by the characteristics $c_{p+1}(1,0)$

and $c_p(1, T)$, corresponding to the slowest negative and positive wave speeds. This decomposition is exhibited in Figure 3.2.

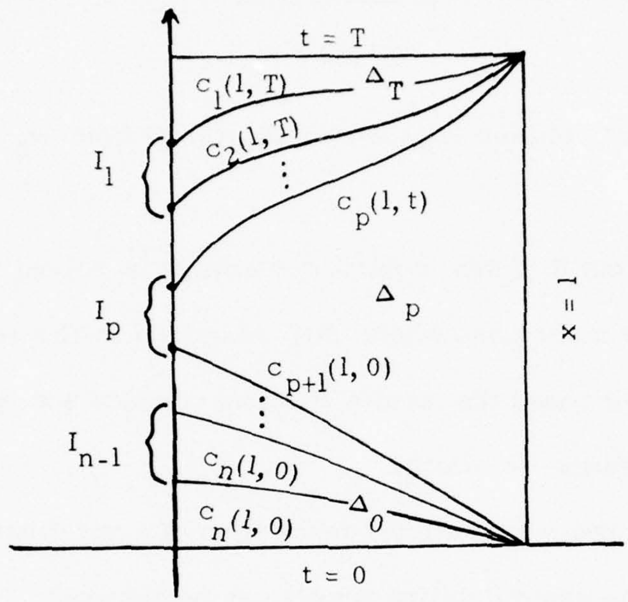

Figure 3.2. Decomposition of $0 \leq x \leq 1$, $0 \leq t \leq T$

We note that there is an upper "triangular" region, Δ_T, above $c_p(1, T)$ and a lower "triangular" region, Δ_0, below $c_{p+1}(1, 0)$. These are separated by a central "trapezoidal" region Δ_p. It is the inequality (3.18) which ensures that $c_p(1, T)$ and $c_{p+1}(1, 0)$ do not meet to the right of the line $x = 0$. All characteristics $c_k(1, 0)$, $p+1 \leq k \leq n$, lie in Δ_0 and all characteristics $c_k(1, T)$, $1 \leq k \leq p$, lie in Δ_T. These circumstances allow solutions w of (3.3) to be constructed in Δ_T from the condition

$$w(\cdot, T) = 0 \tag{3.19}$$

and in Δ_0 from the condition (3.7) i.e. $w(\cdot, 0) = w_0$. For the former we need only set

$$w(x,t) \equiv 0, \quad (x,t) \in \Delta_T .$$

The construction in Δ_T is not quite as quick but still very straightforward. The condition $w(\cdot, 0) = w_0$ together with the boundary condition (3.11) along $x = 0$ can be used together with the integral equation reformulation of (3.6) to construct w in the "domain of determinacy" of $\{t = 0, \; 0 \leq x \leq 1\}$, which is the region between that line segment and the characteristic $c_n(1, 0)$. To proceed further it is then necessary to prescribe (arbitrarily) values for $w^n(0,t)$ on the segment I_{n-1} of $x = 0$ cut out by $c_n(1, 0)$ and $c_{n-1}(1, 0)$. When this has been done w is determined in the region between these two characteristics. At the next stage arbitrary values are assigned to both w^n and w^{n-1} on I_{n-2}, the segment of $x = 0$ cut off by $c_{n-1}(1, 0)$ and $c_{n-2}(1, 0)$ and so is then determined in the region between this next pair of characteristics. This process continues until w has been defined in all of Δ_0. Finally, arbitrary values are assigned to $w^{p+1}, w^{p+2}, \ldots, w^n$ on I_p, the segment of $x = 0$ cut off by $c_{p+1}(1, 0)$ and $c_p(1, T)$ (which has zero length if equality holds in (3.18)). This gives w^+ on I_p and (3.11) then determines w^- there, so that the whole vector w is now known on I_p. The extension of w from $I_p \cup \Delta_0 \cup \Delta_T$ into Δ_p is essentially the Goursat problem, or characteristic initial value problem [11], [31]. Once this extension is completed we have the values of $w(1,t)$,

$0 \le t \le T$, on $x = 1$ (and hence $w^-(1,t)$, $w^+(1,t)$ there). Then (3.13) determines $u(t)$. The basic uniqueness results show that with u thus obtained, the constructed solution w is the unique solution of (3.3), (3.7), (3.11), (3.13) for $0 \le x \le 1$, $0 \le t \le T$ and we clearly have (3.19), completing the first part of the theorem. If $S(t)$ is a group, the requirements for which we have elaborated above, the process can be reversed to obtain a control steering backward from an arbitrary terminal state $w(\cdot, T) = w_1 \in L^2([0,1]; E^n)$ to $w(\cdot, 0) = 0$. The general control problem can then be solved by adding the two solutions just described. It is also possible to do the whole job at once by carrying out the construction which we described in Δ_0 in both Δ_0 and Δ_T when $S(t)$ is a group. This completes the sketch of the proof of the theorem. For further detail, see [86], [90], [11], [31]. We make one further remark, however. If one takes care in assigning the arbitrary data on the intervals I_k, $k = 1, 2, \ldots, n-1$, e.g. if one always sets such data equal to zero, one can show (see e.g. [W]) that

$$\|u\|_{L^2([0,T]; E^q)} \le K_0 \|w_0\|_{L^2([0,1]; E^n)} + K_1 \|w_1\|_{L^2([0,1]; E^n)} \tag{3.20}$$

for some $K_0, K_1 > 0$.

While the above argument perhaps best illustrates the precise role played by the characteristics c_k and the precise degree to which u is

unique or has arbitrary elements, it is not the preferred constructive method. A variant of the above procedure developed by Cirina [9] in a nonlinear context uses a sequence of three successive solutions of initial – boundary value problems in standard rectangular regions – obviously far more convenient for numerical purposes than our extensions into curvilinear triangular regions. The inequality (3.20) is also most easily proved in the context of his procedure. We shall have more to say about Cirina's paper in Section 9.

If

$$T < \int_0^1 \frac{dx}{\lambda_n(x)} - \int_0^1 \frac{dx}{\lambda_1(x)}$$

it is rather easy to see that the control problem cannot be solved in general, even for $w_1 = 0$. In this case the domains of determinacy of the line segments $t = 0$, $0 \leq x \leq 1$, and $t = T$, $0 \leq x \leq 1$, have an intersection with non-empty interior and the two determinations of w in this intersection, the one via $w(x, 0) = w_0(x)$, $0 \leq x \leq 1$, and the other via $w(x, T) = w_1(x)$, $0 \leq x \leq 1$, are in general inconsistent. This is clearly a rather minimal result and we expect non-controllability to hold in many cases where T is larger than this. To study problems of this sort and other control questions of a more complicated nature, one is led to adopt a more indirect approach.

We find it useful to adopt the general abstract scheme set forth in Section 2. We take the spaces X and Z both to be $L^2([0,1]; E^n)$

and we take Y to be $L^2([0,T]; E^q)$ for whatever value of T is under consideration. Choosing for the present to specialize in the control problem with $w_1 = 0$ we define $S: X \to Z$ by solving (3.3), (3.7), (3.11), (3.12), thereby obtaining a solution $\tilde{w}(x,t)$, $0 \le x \le 1$, $0 \le t \le T$, and setting

$$Sw_0 = \tilde{w}(\cdot, T) \quad . \tag{3.21}$$

(Thus $S = S(T)$, the latter having been defined in (3.15).) The operator $C: Y \to Z$, on the other hand, is obtained by solving (3.3), (3.11), (3.13) with $\hat{w}(\cdot, 0) = 0$, thereby obtaining a solution $\hat{w}(x,t)$, $0 \le x \le 1$, $0 \le t \le T$, which depends on u, and setting

$$Cu = \hat{w}(\cdot, T) \quad . \tag{3.22}$$

(Thus $C = C(T)$ as in (3.15).)

From (3.15), (3.16) we see that S and C are both bounded operators in this case.

Now consider the adjoint system

$$\frac{\partial v}{\partial t} = A(x)\frac{\partial v}{\partial x} - (B(x)^* - A'(x))v \tag{3.23}$$

with the adjoint boundary conditions (see [V] for derivation)

$$v^+(0,t) = -(A^+(0))^{-1} D_0^* A^-(0)\, v^-(0,t) \quad , \tag{3.24}$$

$$v^-(1,t) = -(A^-(1))^{-1} D_1^* A^+(1)\, v^+(1,t) \quad . \tag{3.25}$$

All of the results of Theorem 3.1 apply to (3.23), (3.24), (3.25) provided we specify terminal data (the relevant semigroup is defined for $t \le 0$)

$$v(\cdot, T) = v_1 \in L^2([0,1]; E^n) \quad . \tag{3.26}$$

Continuing with the general scheme of Section 2, we define $S^*: Z \to X$ by

$$S^* v_1 = v(\cdot, 0) \in X = L^2([0,1]; E^n) \quad , \tag{3.27}$$

where v is the solution of (3.23) - (3.26), and we define $C^*: Z \to Y = L^2([0,T]; E^q)$ by

$$\omega = C^* v_1 = D^* A^+(1)\, v^+(1, \cdot) \in Y \quad . \tag{3.28}$$

A rather standard computation ([91]) shows that if v satisfies (3.23), (3.24), (3.25) while w satisfies (3.3), (3.11), (3.13) then

$$(v(\cdot, T),\ w(\cdot, T))_{L^2([0,1]; E^n)} - (v(\cdot, 0),\ w(\cdot, 0))_{L^2([0,1]; E^n)}$$

$$= (D^* A^+(1)\, v^+(1, \cdot),\ u)_{L^2([0,T]; E^q)} \quad .$$

For $u = 0$ we have (cf.(3.21), (3.27))

$$(v_1, Sw_0)_{L^2([0,1]; E^n)} - (S^* v_1, w_0)_{L^2([0,1]; E^n)} = 0$$

and for $w(\cdot, 0) = w_0 = 0$ we have (cf. (3.22), (3.28))

$$(v_1, Cu)_{L^2([0,1]; E^n)} = (C^* v_1, u)_{L^2([0,T]; E^q)} \quad ,$$

confirming that S^* and C^* are the adjoint operator to S and C, as anticipated by the notation.

We are now in a position to analyze the control problem for the system (3.3), (3.11), (3.13) with the aid of the duality results embodied in Theorems 2.6 and 2.7 of the preceding section. According to those theorems we may study the control problem, in this case that of steering from $w_0 \in L^2([0,1]; E^n)$ to $w_1 = 0$, by enquiring whether $\{C^*, S^*, Z, Y, X\}$ is distinguishable or observable, respectively, i.e., whether (since $D^* A^+(1)$ is nonsingular in (3.28))

$$v^+(1,t) = 0, \quad t \in [0,T] \Rightarrow v(x,0) = 0, \quad x \in [0,1] \tag{3.29}$$

or, for some $K > 0$,

$$\|v^+(1,\cdot)\|_{L^2([0,T]; E^q)} \geq K \|v(\cdot,0)\|_{L^2([0,1]; E^n)} , \tag{3.30}$$

respectively.

We will first consider the case wherein

$$B^*(x) = A'(x), \quad x \in [0,1] , \tag{3.31}$$

which, of course, includes the case where A is constant and $B = 0$. Defining the characteristics $c_k(x,t)$ as before we see that for $(x(t),t)$ on a characteristic c_k we have

$$\frac{d}{dt} v^k(x(t),t) = \frac{\partial v^k}{\partial t}(x(t),t) - \lambda_k(x(t)) \frac{\partial v^k}{\partial x}(x(t),t) = 0 ,$$

since (3.23) gives $\frac{\partial v}{\partial t} - A(x)\frac{\partial v}{\partial x} = 0$, and $A(x) = \mathrm{diag}(\lambda_1(x), \ldots, \lambda_n(x))$. This means that $v^k(x(t),t)$ is constant on a characteristic c_k, thus greatly simplifying the behavior of the solution v.

Proposition 3.3. *For* $i = p+1, p+2, \ldots, p+q$ *let* T_i^+ *be the unique positive number such that the solution* $x(t)$ *of*

$$\frac{dx}{dt} = -\lambda_i(x), \quad x(0) = 1$$

satisfies $x(T_i^+) = 0$, *and for* $j = 1, 2, \ldots, p$ *let* $x(T_j^-) = 1$. *For* $j = 1, 2, \ldots, p$ *let* i_j *be the largest integer in the set* $\{p+1, \ldots, p+q\}$ *such that the component* $(D_0^*)_j^{i_j}$ *of the matrix* D_0^* (cf. (3.24)) *is non-zero if such an* i_j *exists. Define*

$$\tilde{T}_j = \begin{cases} T_{i_j}^+ + T_j^-, & i_j \text{ exists}, \\ \\ T_j^- & \text{otherwise}, \end{cases}$$

and let

$$T_0 = \max_{\substack{i \in \{p+1, \ldots, p+q\} \\ j \in \{1, 2, \ldots, p\}}} \{T_i^+, \ \tilde{T}_j\} .$$

Assuming (3.31), *if* $T < T_0$ *the distinguishability property* (3.29) (*and hence the observability property* (3.30) *also*) *cannot hold.*

The basic idea here is the construction of paths whereby "information" can be sent from the terminal state $v_1 = v(\cdot, T)$ to the initial state $v_0 = v(\cdot, 0)$ without encountering the boundary $x = 1$ where the observation $D^* A^+(1) v^+(1, t)$ is taken. We will analyze only the case where, for some j_0 and i_{j_0}, we have $T_0 = T_{i_{j_0}}^+ + T_{j_0}^-$. Referring to Fig. 3.3 we note that if $T < T_0$

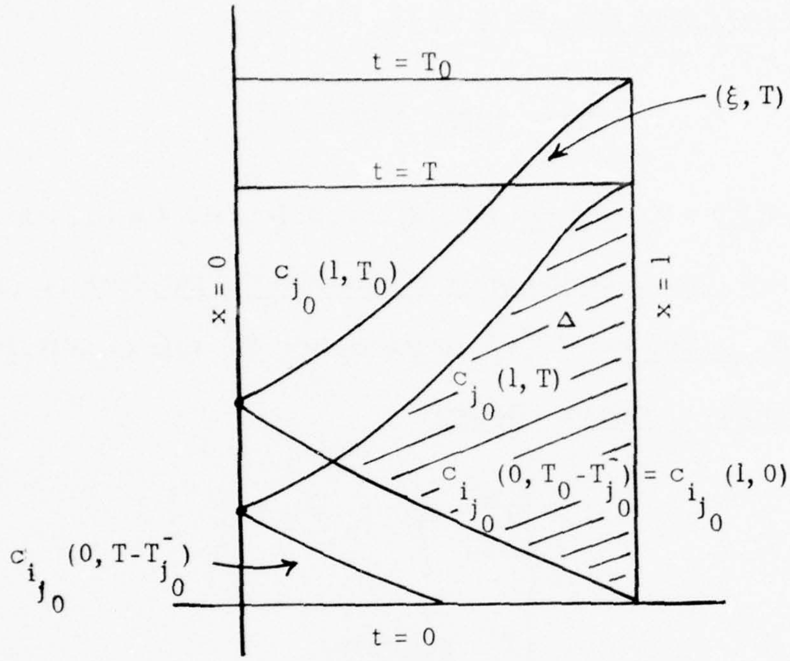

Figure 3.3. Diagram for $T < T_0$

the characteristic $c_{j_0}(1, T_0)$ meets the line $t = T$ at a point (ξ, T), $\xi < 1$. We define a terminal state $v(\cdot, T) = v_1$ by

$$v_1^k(x) = \begin{cases} 0 \text{ if } k \neq j_0 \text{ or if } k = j_0 \text{ and} \\ x \notin (\xi, 1]; \\ 1 \text{ if } k = j_0 \text{ and } x \in (\xi, 1]. \end{cases}$$

Since v^k is constant on characteristics c_k we see that for $t \geq T - T_{j_0}^-$ the only non-zero part of the solution v is the component v^{j_0}, which equals 1 in the strip between $c_{j_0}(1, T_0)$ and $c_{j_0}(1, T)$ (and is equal to zero outside that strip). At the boundary $x = 0$ we have

$$v^k(0,t) \equiv 0, \quad t \geq T_0 - T_{j_0}^-, \qquad k = 1, 2, \ldots, n$$

$$v^k(0,t) \equiv 0, \quad t \geq T \;\; - T_{j_0}^-, \qquad k = 1, 2, \ldots, p, \quad k \neq j_0$$

$$v^{j_0}(0,t) \equiv 1, \quad T - T_{j_0}^- < t < T_0 - T_{j_0}^- \; .$$

We now consider components v^k, $k \in \{p+1, \ldots, p+q = n\}$ related to v^{j_0} at $x = 0$ by the boundary condition (3.24). Since $A(0)$ is diagonal the precautions taken in defining the $\tilde{T}_j$ show that those v^k related to a characteristic $c_k(0,s)$, $T - T_{j_0}^- \leq s < T_0 - T_{j_0}^-$, which could meet the segment $x = 1$, $0 \leq t \leq T_0$, are related to v^{j_0} by a zero coefficient in the matrix $(A^+(0))^{-1} D_0^* A^-(0)$. This enables us to conclude

$$v^+(1,t) \equiv 0, \quad t \in [0,T] \; .$$

On the other hand we shall have

$$v^{i_{j_0}}(x,t) \equiv (A^+(0)^{-1} D_0^* A^-(0))_{j_0}^{i_{j_0}}$$

between the characteristics $c_{i_{j_0}}(0, T_0 - T_{j_0}^-) = c_{i_{j_0}}(1,0)$ and $c_{i_{j_0}}(0, T - T_{j_0}^-)$ so

$$v_0 = v(\cdot, 0) \neq 0$$

and thus (3.29) does not hold. The other possibilities for T_0 are analyzed in a comparable manner.

For a positive result, we have

Proposition 3.4. For $i = p+1, p+2, \ldots, p+q$ let j_i be the largest integer in the set $\{1, 2, \ldots, p\}$ such that the component $(D_0^*)_{j_i}^i$ of the

matrix D_0^* *is non-zero, if it exists. Letting* T_i^+, $i = p+1, \ldots, p+q$, T_j^-, $j = 1, 2, \ldots, p$, *be defined as in Proposition* 3.3, *let*

$$\hat{T}_i = \begin{cases} T_i^+ + T_{j_i}^-, & j_i \ \underline{\text{exists}}, \\ T_i^+ & \underline{\text{otherwise}}, \end{cases}$$

and let

$$T_1 = \max_{\substack{i \in \{p+1, \ldots, p+q\} \\ j \in \{1, 2, \ldots, p\}}} \{\hat{T}_i, \ T_j^-\} . \tag{3.32}$$

If $T \geq T_1$ we have observability in the sense (3.30).

Remark. The time T_0 of Proposition 3.3 does not, in general, agree with the time T_1 defined here. This raises the question, unresolved at the moment, concerning the identification of a "critical time" T_c such that observability holds if $T \geq T_c$ and does not hold if $T < T_c$. Such a critical time T_c can readily be shown to exist but no satisfactory characterization of it is available at this writing.

Sketch of Proof. Suppose that

$$S^* v_1 = v(0, T) \neq 0 .$$

Suppose, then, that $x \in (0, 1)$ and k are such that

$$v^k(x, T) \neq 0 . \tag{3.33}$$

If $k \in \{1, 2, \ldots, p\}$ the characteristic $c_k(x, T)$ $(1, \tau)$, $0 < \tau < T$, and then $v^k(1, \tau) \neq 0$, so that $v^-(1, \tau) \neq 0$. Since (3.25) is satisfied at $(1, \tau)$, we will have

$$v^+(1, \tau) \neq 0 \quad .$$

On the other hand, if $k \in \{p+1, \ldots, p+q\}$ then, since $T \geq T_k^+$ we will have the characteristic $c_k(x, T)$ meeting $x = 0$ at some point τ_1, $0 < \tau_1 < T$ and $v^k(0, \tau_1) \neq 0$. Since (3.24) is satisfied at $x = 0$, some component $(D_0^*)_\ell^k$ of D_0^* is non-zero and

$$v^\ell(0, \tau_1) \neq 0 \quad .$$

Now (3.32) shows that $T_1 \geq T_k^+ + T_\ell^-$, and since $T \geq T_1$, $c_\ell(0, \tau_1)$ meets $x = 1$ at a point $(1, \tau_2)$ with $0 < \tau_2 < T$ and, since v^ℓ is constant on $c_\ell(0, \tau_1)$, we have $v^\ell(1, \tau_2) \neq 0$. Then, as before

$$v^+(1, \tau_2) \neq 0 \quad .$$

In either case, by letting x vary in some non-null subset of $(0, 1)$ where (3.33) holds, we see that $v^+(1, \cdot)$ *is not the null element of* $L^2([0, T]; E^q)$ and distinguishability, (3.29), has been proved. A slightly more careful analysis, keeping track of the norms involved, establishes observability, (3.30).

The duality theory of Section 2 then gives

Corollary 3.5. For the hyperbolic control system (3.3), (3.11), (3.12) there are positive numbers T_0 and T_1 such that the problem of control from an arbitrary $w_0 \in L^2([0,1]; E^n)$ to $w_1 = 0$ during $[0, T]$ is not solvable if $T < T_0$ and is solvable if $T \geq T_1$, provided (3.31) is valid.

Our next task is to see if the restriction $B^*(x) = A'(x)$ can be removed. This is done in two steps. One first considers the case wherein

$B(x) - A'(x)$ is diagonal (i.e., $B(x)$ is diagonal, since $A(x)$ is already in that form). Here the proofs of Proposition 3.3 and 3.4 are virtually unchanged. Although the v^k are not constant on characteristics c_k, they nevertheless satisfy uncoupled scalar linear homogeneous equations

$$\frac{dv^k}{dt}(x(t), t) = (B^T(x) - A'(x))_k^k \, v^k(x(t), t) \quad ,$$

$$k = 1, 2, \ldots, n$$

and hence there are positive β_0, β_1 such that

$$\beta_0 \le \left| \frac{v^k(x(t), t)}{v^k(x(\hat{t}), \hat{t})} \right| \le \beta_1 \tag{3.34}$$

whenever $(x(t), t)$, $(x(\hat{t}), \hat{t})$ lie on the same characteristic c_k. The relationship (3.34) is clearly just as useful in the proofs of Propositions 3.3 and 3.4 as the constancy of v^k on c_k which we assumed there.

The next steps are somewhat more subtle and are in the nature of a perturbation argument. We have to divide our result into two parts.

Theorem 3.6. For the control system (3.3), (3.11), (3.13) and associated dual linear observed system (3.23), (3.24), (3.25), (3.28) we do not have exact controllability to $w_1 = 0$ and observability, respectively, if $T < T_0$. (However, this conclusion is not shown to extend to lack of approximate controllability and distinguishability, respectively.)

Sketch of Proof. Let the operators S and C be defined as in (3.27), (3.28) for the general control system and let the notations S_d and C_d

be employed for the operators related to the corresponding "diagonal" system wherein (3.23) is replaced by

$$\frac{\partial v}{\partial t} = A(x)\frac{\partial v}{\partial x} + (B^*(x) - A'(x))_d\, v \ ,$$

where $(B^*(x) - A'(x))_d$ denotes the diagonal part of that matrix. A somewhat involved, but not conceptually difficult, argument allows one to see that the operator differences $S^* - S_d^*$, $C^* - C_d$ are both compact.

When $T < T_0$, as in Proposition 3.3, a slightly more careful analysis shows that there is an infinite dimensional subspace, call it Z_0, of $L^2([0,1]; E^n)$ such that

$$\|S_d^* v_1\|_{L^2([0,1]; E^n)} \geq K_1 \|v_1\|_{L^2([0,1]; E^n)}, \quad v_1 \in Z_0 \ , \tag{3.35}$$

for some $K_1 > 0$, while

$$C_d^* v_1 = 0, \quad v_1 \in Z_0 \ . \tag{3.36}$$

With (3.35), (3.36) true, it is a standard exercise in functional analysis to show that we cannot have (3.30) if $S^* - S_d^*$ and $C^* - C_d^*$ are both compact. The conclusion that we do not have observability of (3.23) - (3.28), and hence (exact) controllability of (3.3), (3.11), (3.13) for $T < T_0$ follows. Note however that one cannot show that (3.29) is false.

The general controllability result for $T \geq T_1$ is considerably more complicated. Before stating the theorem we present an example to show that this should, indeed, be the case. Consider the two dimensional system

$$\frac{\partial}{\partial t}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}, \quad 0 < x < 1, \quad t \geq 0 \tag{3.37}$$

with boundary conditions

$$v^1(0,t) = 0, \quad v^2(1,t) = 0 \tag{3.38}$$

and observation

$$C^* v_1 = C^* v(\cdot, T) = v^1(1, \cdot) \in L^2[0,1] \ . \tag{3.39}$$

Here one readily verifies that we have observability for $T \geq T_1 = 1$. Indeed, no matter what $v_1 = v(\cdot, 1)$ may be, we have

$$S^* v_1 = v(\cdot, 0) = 0 \ .$$

Thus the observability condition (3.30) is satisfied, as it were, by "default" on the part of S^* rather than by any "strength" of the observation operator (3.39). It is perhaps not surprising, then, that a small perturbation in (3.37) yields a system which is not observable.

Indeed, for $\varepsilon > 0$, let (3.37) be perturbed to

$$\frac{\partial}{\partial t}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix} + \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}, \tag{3.40}$$

with the boundary conditions (3.38) and the observation operator (3.39) being retained. One may then verify quite readily that if we give the terminal state

$$v^1(x,1) \equiv \frac{\varepsilon}{2}(1-x), \quad v^2(x,1) \equiv 1, \quad 0 \leq x \leq 1$$

the effect of the coupling of v^1 to v^2 in (3.40) is to give

$$v^1(1,t) \equiv 0, \quad 0 \leq t \leq 1$$

that $C^* v_1 = 0$. But

$$v^1(x, 0) = v^1(0, x) - \int_{\frac{x}{2}}^{x} \varepsilon \, dt = -\frac{\varepsilon x}{2}$$

so $v^1(x, 0)$, and hence $S^* v_1 = v(\cdot, 0) \neq 0$. It follows that (3.40), (3.38), (3.39) is not even distinguishable on $[0, 1]$, for any $\varepsilon > 0$.

We first attempt to rule out this sort of behavior by strengthening S^*. *This we do by assuming the system* (3.3), (3.11), (3.12) *to be time* reversible, the requirements for which we have set down earlier. Then the operator (3.14) generates a group $S(t)$ for $t \in (-\infty, \infty)$ and S, S^* are invertible. In fact, with S_d as defined earlier, we shall have, for $v_1 \in L^2([0,1]; E^n)$,

$$\|S_d^* v_1\|_{L^2([0,1]; E^n)} \geq M_d \|v_1\|_{L^2([0,1]; E^n)} . \tag{3.41}$$

Theorem 3.7. Let (3.3), (3.11), (3.12) be time reversible (see note in statement of Theorem 3.1) and let $T \geq T_1$ as described in Proposition 3.4. Then there is an $\varepsilon > 0$, depending only on T, $A(x)$, the diagonal elements of $B(x)$, and the matrices C_0, C_1, D (cf. (3.11), (3.12), (3.13)) such that if each off-diagonal element $b_j^i(x)$, $i \neq j$, satisfies $|b_j^i(x)| < \varepsilon$, $x \in [0,1]$, then the system (3.23), (3.24), (3.25), (3.28) remains observable on $[0, T]$ (equivalently (3.3), (3.11), (3.13) remains controllable on $[0, T]$).

Sketch of Proof. Since we have observability for the diagonal system we have

$$\|C_d^* v_1\|_{L^2([0,1];E^q)} \geq K\|S_d^* v_1\|_{L^2([0,1];E^n)}$$

$$\geq KM_d \|v_1\|_{L^2([0,1];E^n)}, \quad v_1 \in L^2([0,1];E^n) . \tag{3.42}$$

Let the terminal state v_1 be given and let $\tilde{v}(x, t)$ be the resulting solution of the diagonal system for $t \leq T$, $v(x, t)$ the corresponding solution of the complete system (3.23), (3.24), (3.25) (with the off diagonal elements of $B(x)$ restricted as above). Then $\hat{v} = v - \tilde{v}$ satisfies the boundary conditions (3.24), (3.25) again, the terminal condition

$$\hat{v}(x, T) = 0, \quad x \in [0,1] \tag{3.43}$$

and the system of differential equations (cf. (3.23))

$$\frac{\partial \hat{v}}{\partial t} = A(x)\frac{\partial \hat{v}}{\partial x} - (B_d^*(x) - A'(x))\hat{v} + (B^*(x) - B_d^*(x))(\hat{v} + \tilde{v}) , \tag{3.44}$$

where $B_d(x)$ denotes the diagonal part of $B(x)$. Constructing and estimating the solution of (3.43), (3.42), (3.24), (3.25), in the process making use of the fact that $\tilde{v}$ can already be bounded in terms of $v_1 = \tilde{v}(x, T)$, the condition on the elements $b_j^i(x)$, $i \neq j$, of $B^*(x) - B_d^*(x)$ shows that

$$\|(S^* - S_d^*)v_1\|_{L^2([0,1];\, E^n)} = \|\hat{v}(\cdot, 0)\|_{L^2([0,1];\, E^n)}$$

$$\leq \hat{K}(\varepsilon) \|v_1\|_{L^2([0,1];E^n)} , \tag{3.45}$$

$$\|(C^* - C_d^*)v_1\|_{L^2([0,1];E^q)} = (\text{cf. } (3.28))\ \|D^* A^+(1)\hat{v}^+(1,\cdot)\|_{L^2([0,1];E^q)}$$

$$\leq \hat{L}(\varepsilon)\|v_1\|_{L^2([0,1];E^n)} \tag{3.46}$$

for positive $\hat{K}(\varepsilon)$, $\hat{L}(\varepsilon)$ tending to zero as $\varepsilon \to 0$. From (3.41), (3.42), (3.45), (3.46) one easily obtains (3.30) for some $M > 0$, provided ε is sufficiently small, and the proof is complete.

We remind the reader that, while Theorem 3.7 is stated in terms of the operator S defined in (3.21), and hence directly yields only the controllability to $w(\cdot, T) = 0$ for (3.3), (3.11), (3.13), the assumption that this system is time reversible guarantees that $\mathcal{R}(S) = \mathcal{R}(I)$ so that controllability to any given terminal state follows immediately.

The condition that the off diagonal elements be small can be relaxed if we assume approximate controllability to hold for the perturbed system, i.e.

$$C^* v_1 = 0 \Rightarrow S^* v_1 = 0 \ . \tag{3.47}$$

Since S, and hence S^*, is boundedly invertible when (3.3), (3.11), (3.12) is time reversible, (3.47) is the same as

$$C^* v_1 = 0 \Rightarrow v_1 = 0 \ .$$

Then (3.42), together with the fact that $C^* - C_d^*$ is compact, as noted earlier, allows the Fredholm alternative theorem [84] to be applied to give

$$\|C^* v_1\|_{L^2([0,1];E^n)} \geq K_1\|v_1\|_{L^2([0,1];E^n)} \geq K_2\|S^* v_1\|_{L^2([0,1];E^n)}$$

for positive K_1, K_2. That is, with a compact perturbation, as we have here, distinguishability implies observability. More detail is given in the proof of a comparable theorem in [16]. However, this result is not too useful because distinguishability is rather easily destroyed. We have seen this to be the case in the example given prior to Theorem 3.7. There, however, the system is not time reversible. To see that it can still happen for a time reversible system we consider a second example.

We form the four dimensional system

$$\frac{\partial}{\partial t}\begin{pmatrix} v_1^- \\ v_2^- \\ v_1^+ \\ v_2^+ \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} v_1^- \\ v_2^- \\ v_1^+ \\ v_2^+ \end{pmatrix} + \begin{pmatrix} 0 & \alpha & 0 & \beta & v_1^- \\ 0 & 0 & 0 & 0 & v_2^- \\ 0 & -\beta & 0 & -\alpha & v_1^- \\ 0 & 0 & 0 & 0 & v_2^+ \end{pmatrix}, \quad (3.48)$$

with α, β yet to be determined, and boundary conditions

$$v_1^+(0,t) = v_2^-(0,t), \;\; v_2^+(0,t) = v_1^-(0,t) \;\;, \quad (3.49)$$

$$v_1^-(1,t) = v_2^+(1,t), \;\; v_2^-(1,t) = v_1^+(1,t) \;\;. \quad (3.50)$$

The observation is

$$C^* v(\cdot, T) = \begin{pmatrix} (v_1^+(1,\cdot) \\ \\ (v_2^+(1,\cdot) \end{pmatrix} \quad . \quad (3.51)$$

Clearly we have $p = q = 2$ and (3.49), (3.50) shows that (cf. (3.24), (3.25)) we must have

$$-(A^+(0))^{-1} D_0^* A^-(0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad -(A^-(1))^{-1} D_1^T A^+(1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which are invertible, so (3.48), (3.49), (3.50) is time reversible. For $\alpha = \beta = 0$, Proposition 3.4 shows the system to be observable for $T \geq T_1 = 3$. But for $T = T_1 = 3$ and

$$\alpha = -1, \ \beta = -3 \tag{3.52}$$

the system is not even distinguishable. This is the situation diagrammed in Figure 3.4.

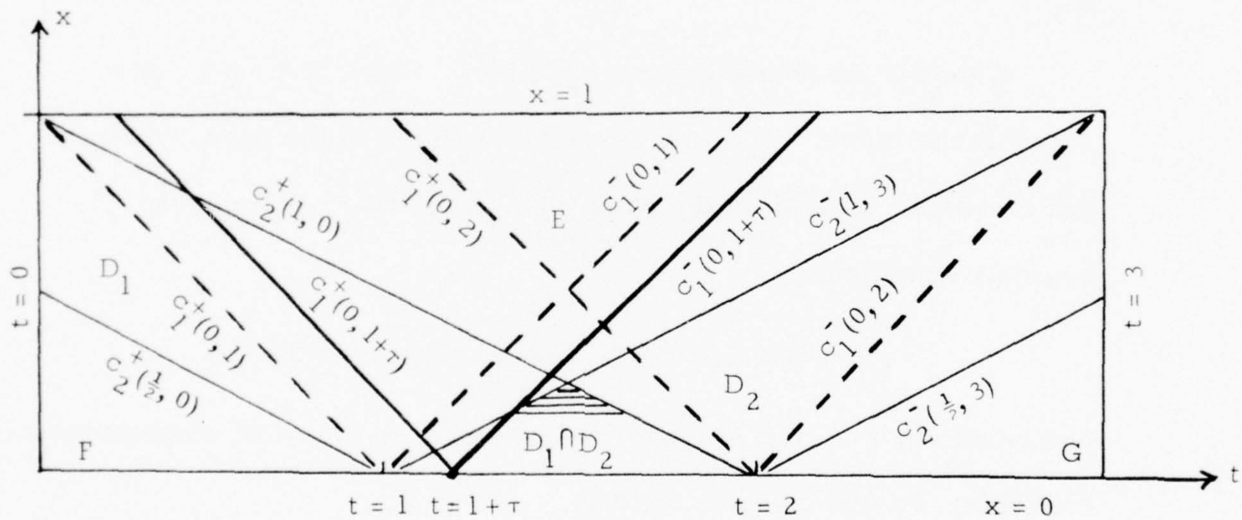

Figure 3.4. The region $0 \leq t \leq 3$, $0 \leq x \leq 1$

We begin by defining (see Fig. 3.4)

$$v_2^+(x, t) \equiv 1, \quad (x, t) \in D_1, \quad v_2^-(x, t) \equiv 1, \quad (x, t) \in D_2 \tag{3.53}$$

in the shaded regions D_1, D_2 shown in Fig. 3.4, bounded by the

characteristics $c_2^+(1, 0)$, $c_2^+(\frac{1}{2}, 0)$, $c_2^-(1, 3)$, $c_2^-(\frac{1}{2}, 3)$, the segment of the t axis between $t = 1$ and $t = 2$ and the segments of $t = 0$, $t = 3$ between $x = \frac{1}{2}$ and $x = 1$. Further we let

$$v_2^+(x, t) \equiv 0, \quad (x, t) \text{ lies above } c_2^+(1, 0) \text{ in Fig. 3.4}, \tag{3.54}$$

$$v_2^-(x, t) \equiv 0, \quad (x, t) \text{ lies above } c_2^-(1, 3) \text{ in Fig. 3.4}. \tag{3.55}$$

$$v_2^+(1, t) \equiv v_2^-(1, t) \equiv 0, \quad 0 \leq t \leq 3 \tag{3.56}$$

and leaves v_2^+ undefined only in region F, v_2^- undefined only in region G.

Consider the characteristic $c_1^-(0, 1+\tau)$, where $0 \leq \tau \leq 1$, as shown in the figure. This line intersects $c_2^-(1, 3)$ at the point $(\tau, 1+2\tau)$ and intersects $c_2^+(1, 0)$ at the point $(\frac{1}{3} - \frac{1}{3}\tau, \frac{4}{3} + \frac{2\tau}{3})$. Using the boundary conditions

$$v_1^-(0, 1+\tau) = v_2^+(0, 1+\tau) \; (\equiv 1 \text{ from (3.53)}, \text{ if } 0 \leq \tau < 1)$$

and integrating the first equation implied by (3.48) along the characteristic $c_1^-(0, 1+\tau)$ we find using (3.52) that

$$\begin{aligned} v_1^-(1, 2+\tau) &= 1 + \alpha\tau + \beta(\frac{1}{3} - \frac{1}{3}\tau) \\ &= 1 - \tau - 3(\frac{1}{3} - \frac{1}{3}\tau) \equiv 0, \quad 0 \leq \tau \leq 1 . \end{aligned} \tag{3.57}$$

On similar analysis using the characteristic $c_1^+(0, 1+\tau)$ gives

$$v_1^+(1, \tau) \equiv 0, \quad 0 \leq \tau \leq 1 . \tag{3.58}$$

For $t = 0$, $0 \leq x < \frac{1}{2}$ we put

$$v_2^+(x, 0) = 1 \ ,$$

which gives $v_2^+(x, t) \equiv 1$ in region F. For $t = 3$, $0 \leq x < \frac{1}{2}$ we put

$$v_2^-(x, 3) = 1$$

which gives $v_2^-(x, t) \equiv 1$ in region G. Integrating over characteristics $c_1^-(0, \tau)$, $c_1^+(0, 2+\tau)$ one find that

$$v_1^-(1, 1+\tau) \equiv 0, \quad 0 \leq \tau < 1 \tag{3.59}$$

$$v_1^+(1, 2+\tau) \equiv 0, \quad 0 \leq \tau \leq 1 \ . \tag{3.60}$$

Defining

$$v_1^-(x, 0) = 1, \quad 0 \leq x < \tfrac{1}{2} \ ,$$
$$v_1^+(x, 3) = 1, \quad 0 \leq x < \tfrac{1}{2} \ ,$$
$$v_1^-(x, 0) = 2(1-x), \quad \tfrac{1}{2} \leq x < 1 \ ,$$
$$v_1^+(x, 3) = 2(1-x), \quad \tfrac{1}{2} \leq x < 1 \ ,$$

and integrating over characteristics $c_1^-(x, 0)$, $c_1^+(x, 3)$ one has

$$v_1^-(1, x) \equiv 0, \quad 0 \leq x < 1 \ , \tag{3.61}$$

$$v_1^+(1, 3-x) \equiv 0, \quad 0 \leq x < 1 \ . \tag{3.62}$$

Finally we note that (3.49), (3.50), (3.54), (3.55) show that $v_1^+(x, t) \equiv 0$ below $c_1^+(0, 1)$ and $v_1^-(x, t) \equiv 0$ below $c_1^-(0, 2)$. All quantities have now been defined consistent with (3.48) and (3.49), (3.50). The solution is clearly not identically equal to zero at $t = T = 3$ but (3.54)-(3.62) show that $C^* v(\cdot, 3)$ (cf. (3.51)) is the zero element of $L^2([0, 3]; E^2)$. We

conclude that (3.48), (3.49), (3.50), (3.51) is not observable in [0, 3] when α and β are given by (3.52).

To summarize: as seen in the example before Theorem 3.7, when the system (3.3), (3.11), (3.12) is not time reversible it is possible that observability, as defined in terms of (3.24), (3.25), (3.26), (3.28), can be destroyed by arbitrarily small perturbations in the off-diagonal elements of the matrix $B^*(x)$. Theorem 3.7 shows this cannot happen if the system is time reversible, the perturbations then must be suitably large to destroy observability. The example immediately preceding this paragraph shows that observability and even distinguishability can be destroyed by "suitably large" perturbations for $T = T_1$.

Now we come to a rather curious punch line. The fragility of observability of (3.24), (3.25), (3.26), (3.28) (and hence of controllability of (3.3), (3.11), (3.13)) exhibited in these examples for $T = T_1$ (=1 in the first example, =3 in the second example) does not occur if we use the control time (3.18) of Theorem 3.2. The constructive proof of controllability described there is in no way affected by any changes in the matrix $B(x)$ – it works for every continuous $B(x)$. Thus in the example before Theorem 3.7 we cannot destroy observability via $B(x)$ if we take $T = 2$, and in the example following that theorem we cannot destroy observability via $B(x)$ if we take $T = 4$. In other words, the observability and controllability associated with the "overly generous" time interval defined in Theorem 3.2 are more "robust" than those associated with the generally

more precise time interval defined in Proposition 3.4 and Corollary 3.5. This casts some doubt on the usefulness of these latter results in practice and makes the search for the critical time T_c, described after the statement of Proposition 3.4, of rather academic (if any) interest.

In Section 2 we have noted the close connection between controllability and stabilizability which obtains in the case of finite dimensional linear control systems. We now proceed to show that, at least in part, this relationship can be extended to the infinite dimensional control systems which we have been discussing here. For the general system (3.3), (3.11), (3.12) our results are quite incomplete. A special case, to be discussed in Section 4, can be treated much more satisfactorily.

The class of systems which we shall treat is that described by (3.3), (3.11), (3.12) with the additional

Assumption 3.8. For each $w \in E^n$ and $x \in [0,1]$

$$(w, (B^*(x) + B(x) - A'(x))w)_{E^n} \leq 0 \ . \tag{3.63}$$

Further, for each vector $w^- \in E^p$, $w^+ \in E^q$, and some $\gamma > 0$,

$$(w^-, (A^-(1) + D_1^* A^+(1) D_1) w^-)_{E^p} \leq -\gamma \| w^- \|^2_{E^p} \tag{3.64}$$

$$(w^+, (D_0^* A^-(0) D_0 + A^+(0)) w^+)_{E^q} \leq 0 \ . \tag{3.65}$$

For continuously differentiable solutions of (3.3), (3.11), (3.12) (which correspond to initial states $w(\cdot, 0) = w_0$ in the domain of the generator, L, of the semigroup $S(t)$ described in Theorem 3.1) we can study the evolution of the norm of the solution $w(\cdot, t)$ by computing

$$\frac{d}{dt}\|w(\cdot,t)\|^2_{L^2([0,1];E^n)} = \frac{d}{dt}\int_0^1 \|w(x,t)\|^2_{E^n}\,dx$$

$$= \int_0^1 [(\frac{\partial w}{\partial x}(x,t),\ A(x)w(x,t))_{E^n}$$

$$+(w(x,t),\ A(x)\frac{\partial w}{\partial x}(x,t))_{E^n} + (w(x,t),\ (B^*(x) + B(x))\ w(x,t))_{E^n}]dx$$

(integrating the first term by parts) =

$$= \int_0^1 (w(x,t),\ (B^*(x) + B(x) - A'(x))\ w(x,t))_{E^n}\ dx$$

$$+(w(1,t),\ A(1)\ w(1,t))_{E^n} - (w(0,t),\ A(0)\ w(0,t))_{E^n}\ .$$

Using the first part of Assumption 3.8 and the boundary conditions (3.11) and (3.12) we now have

$$\frac{d}{dt}\|w(\cdot,t)\|^2_{L^2([0,1];E^n)} \le (w^-(1,t),\ (A^-(1) + D_1^*\ A^+(1)\,D_1)w^-(1,t))_{E^p}$$

$$-(w^+(0,t),\ (D_0^*\,A^-(0)\,D_0 + A^+(0))\,w^+(0,t))_{E^q} \le \text{(using (3.64), (3.65))}$$

$$-\gamma\|w^-(1,t)\|^2_{E^p}\ .$$

The integrated form of this inequality, i.e.,

$$\|w(\cdot,0)\|^2_{L^2([0,1];E^n)} - \|w(\cdot,T)\|^2_{L^2([0,1];E^n)} \ge \gamma\int_0^T \|w^-(1,t)\|^2_{E^p}\,dt$$

$$(= \gamma\|w^-(1,\cdot)\|^2_{L^2([0,T];E^p)} \qquad (3.66)$$

remains true for generalized solutions of (3.3), (3.11), (3.12) corresponding to arbitrary initial states $w(\cdot, 0) \in L^2([0,1]; E^n)$.

In physical applications the inequality (3.63) of Assumption 3.8 states that there is no energy source in the interior of the medium under consideration. Likewise (3.65) states that there is no energy source at the boundary $x = 0$. The inequality (3.64) is stronger, implying that there is an actual loss of energy at $x = 1$ which is greater than or equal to a certain positive fraction of the norm of the "incoming information" at $x = 1$ at the given instant t. We remark that (3.64) and (3.65) are true, respectively, if D_1 and D_0, respectively, are sufficiently small. In some cases this might not be true a priori. One might then augment (3.12) to (3.13) by use of a control variable:

$$w^+(1, t) = D_1 w^-(1, t) + Du(t) \ .$$

A boundary "feedback" relation

$$u(t) = K w^-(1, t) \ , \tag{3.67}$$

expressing the control $u(t)$ in terms of the "incoming information" at $x = 1$ at the instant t could then be employed to give

$$w^+(1, t) = \hat{D}_1 w^-(1, t) = (D_1 + DK) w^-(1, t) \ .$$

Since D is assumed nonsingular, K could be selected so that $\hat{D}_1$ is small and, consequently, (3.64) holds.

One suspects from (3.66) that

$$\lim_{t \to \infty} \|w(\cdot, t)\|_{L^2([0,1]; E^n)} = 0 \quad . \tag{3.68}$$

This is in fact true and even more can be established; namely, that there is a uniform exponential decay rate for the norm of the solution. The quantity $\|w(\cdot, t)\|^2_{L^2([0,1];E^n)}$ is being used here as a Liaponnov function and the inequality (3.66) guarantees that it is non-increasing and, in fact, decreasing whenever $w^-(1,t) \neq 0$. The situation is quite analogous that encountered in the study of the damped harmonic oscillator

$$\ddot{x} + \dot{x} + x = 0 \quad . \tag{3.69}$$

The energy, $\frac{1}{2}[(\dot{x})^2 + (x)^2]$, here obeys

$$\frac{d}{dt}\left(\tfrac{1}{2}[(\dot{x})^2 + (x)^2]\right) = -(\dot{x})^2$$

and decreases when $\dot{x} \neq 0$. Equation (3.69) is usually analyzed by recourse to the so-called "invariant set principle" ([52]) which states that the solution tends to the largest subset of $\{(x, \dot{x}) \mid \dot{x} = 0\}$ which is invariant under (3.69). Then one shows that $\dot{x}$ cannot vanish on any interval so this invariant set must be the origin.

In the case of (3.66) we have a rather novel situation in that $w^-(1,t)$ may vanish on an interval of positive length. In fact, if the support of $w(\cdot, 0)$ is restricted to $[0, \varepsilon]$ then we shall have $w(1,t)$ (and hence $w^-(1,t)) \equiv 0$ for $t \in [0, \tau)$,

$$\tau = -\int_{\varepsilon}^{1} \frac{dx}{\lambda_1(x)} \quad .$$

It would nevertheless, be possible to use the extended invariant set

principle ([12], [34] and applications in [94], [105] to establish (3.68). We will indicate how this is done in a slightly different context in Section 6. However this approach by itself would only yield (3.68) and not an estimate of the rate of decay. Consequently we elect to go via a slightly different route (but which, in the end, amounts to more or less the same method).

What we will obtain is an estimate

$$\int_0^T \|w^-(1,t)\|^2_{E^p}\, dt \geq \beta \|w(\cdot, T)\|^2_{L^2([0,1];E^n)} ,$$

which, in the light of the earlier work of this section, one immediately recognizes as an observability type result, $w^-(1,\cdot)$ being the observation. In [82], where exponential decay of solutions of (3.3), (3.11), (3.12) under the hypotheses of Assumption 3.8 was first (to our knowledge) obtained, the authors, Rauch and Taylor, obtain this "observability" result directly. Because we already have Theorem 3.2, we elect to proceed by a "dual" method making use of controllability.

Consider the dual system (3.23), (3.24), (3.25), but augmented with a control vector $f \in E^p$ so that (3.25) is replaced by

$$v^-(1,t) = -(A^-(1))^{-1} D_1^* A^+(1)\, v^+(1,t) + f(t) . \qquad (3.70)$$

Now, as noted earlier, solutions of (3.23), (3.24), (3.70) are defined corresponding to specified terminal data. The control problem which we pose, then, is to steer the terminal state

$$v(\cdot, T) = v_1 \in L^2([0,1]; E^n) \tag{3.71}$$

to the initial state

$$v(\cdot, 0) = 0 \tag{3.72}$$

by use of the control $f \in L^2([0, T]; E^p)$. Taking

$$T \geq \int_0^1 \frac{dx}{\lambda_{p+1}(x)} - \int_0^1 \frac{dx}{\lambda_p(x)}$$

as before, Theorem 3.2 applies (with the time sense reversed) to show that this control problem has a solution. Moreover, the estimate (3.20) applies; translated into our setting it reads

$$\|f\|_{L^2([0,T];E^p)} \leq K_0 \|v(\cdot, T)\|_{L^2([0,1];E^n)} . \tag{3.73}$$

The idea now is to make a particularly advantageous choice of $v(\cdot, T)$. Let w be the solution of (3.3), (3.11), (3.12) under consideration, for which we have developed the inequality (3.66), and with T satisfying (3.18), let

$$v(\cdot, T) = v_1 = w(\cdot, T) . \tag{3.74}$$

Letting f be the control described above, but for this particular terminal state, we use a computation quite similar to that used to obtain (3.66) (but now involving $\frac{d}{dt}(w(\cdot, t), v(\cdot, t))_{L^2([0,1]; E^n)}$) to obtain

$$\|w(\cdot\,,T)\|^2_{L^2([0,1];E^n)} = (\text{from } (3.74))\ (w(\cdot\,,T),\ v(\cdot\,,T))_{L^2([0,1];E^n)}$$

$$-(w(\cdot\,,0),\ v(\cdot\,,0))_{L^2([0,1];E^n)} = \int_0^T (w^-(1,t)\,A^-(1),\ f(t))_{E^p}\,dt\ .$$

$$\leq \text{(Schwarz inequality)}\ \alpha\|w^-(1,\cdot)\|_{L^2([0,T];E^p)}\ \|f\|_{L^2([0,T];E^p)}\ ,$$

with $\alpha > 0$ depending only on T, $A^-(1)$. Then from (3.73) we have

$$\|w^-(1,\cdot)\|_{L^2([0,T];E^p)} \geq \frac{\|w(\cdot\,,T)\|_{L^2([0,1];E^n)}}{\alpha K_0}\ .$$

Then (3.66) gives

$$\|w(\cdot\,,0)\|^2_{L^2([0,1];E^n)} - \|w(\cdot\,,T)\|^2_{L^2([0,1];E^n)} \geq \frac{\gamma}{\alpha^2 K_0^2}\ \|w(\cdot\,,T)\|^2_{L^2([0,1];E^n)}$$

so that

$$\|w(\cdot\,,T)\|^2_{L^2([0,1];E^n)} \leq \frac{1}{1+\gamma/\alpha^2 K_0^2}\ \|w(\cdot\,,0)\|^2_{L^2([0,1];E^n)}\ .$$

Since the coefficient matrices in (3. 3), (3. 11), (3. 12) do not depend on t, the above argument may be repeated on successive intervals $[kT,\ (k+1)T]$, $k = 1, 2, 3, \ldots$ to give

$$\|w(\cdot\,,(k+1)T)\|^2_{L^2([0,1];E^n)} \leq \frac{1}{1+\gamma/\alpha^2 K_0^2}\ \|w(\cdot\,,kT)\|^2_{L^2([0,1];E^n)}$$

whence

$$\|w(\cdot, kT)\|^2_{L^2([0,1];E^n)} \leq \left(\frac{1}{1+\gamma/\alpha^2 K_0^2}\right)^k \|w(\cdot, 0)\|^2_{L^2([0,1];E^n)} .$$

Since (3.66) shows the norm of $w(\cdot, t)$ to be non-increasing with t, we have, for some $M > 0$ and $\mu = T^{-1} \log(1 + \gamma/\alpha^2 K_0^2)$,

Theorem 3.9. *Let* $w = w(x, t)$ *be a generalized solution of* (3.3), (3.11), (3.12) *and let Assumption* 3.8 *be satisfied. Then, for* $t \geq 0$,

$$\|w(\cdot, t)\|^2_{L^2([0,1];E^n)} \leq Me^{-\mu t} \|w(\cdot, 0)\|^2_{L^2([0,1];E^n)} ,$$

i.e., the semigroup $S(t)$ *described in Theorem* 3.1 *has the property*

$$\|S(t)\| \leq M^{\frac{1}{2}} e^{-\frac{1}{2}\mu t}, \quad t \geq 0 .$$

This theorem has some features in common with Theorem 2.9. In fact one can give a proof that the feedback relation (2.21) stabilizes the finite dimensional control system (2.1) which exactly parallels the proof given above for Theorem 3.9. But we are far from a "controllability implies stabilizability" type result. In [103], [105] Slemrod has given an extension of the Lukes-Kleinman stabilization method to a wide class of (exactly) controllable systems in Hilbert space. That method can be extended to cover the present case also but its development in the context of the present boundary control system would be much too lengthy and technical for presentation here. In effect, Slemrod's result would provide a formula comparable to (2.21). In the present situation, assuming

controllability of (3.3), (3.11), (3.13) on some interval $[0, T]$, but not requiring Assumption 3.8, that method would provide a feedback law (cf. (3.67)) of the form

$$u(t) = K w^-(1, t) + \int_0^1 L(x)\, w(x, t) dx$$

for which the closed loop system consisting of (3.3), (3.11) and (cf. (3.12))

$$w^+(1, t) = (D_1 + DK)\, w^-(1, t) + D \int_0^1 L(x)\, w(x, t) dx$$

would have uniform exponential decay for $\|w(\cdot, t)\|_{L^2([0,1];E^n)}$ as $t \to \infty$. In [90], by a different procedure, precisely this result has been obtained for a two dimensional system of the form (4.1). However, a much more complete theory, comparable to the last statement of Theorem 2.9 has recently become available for these simple two dimensional systems and will form the basis for our discussion in Section 4.

4. CONTROLLABILITY VIA HARMONIC ANALYSIS, CONTROL CANONICAL FORM, SPECTRAL DETERMINATION

Here we consider a very special subclass of the hyperbolic systems treated in Section 3 and, at least at the outset, we postulate a different form of control action from that of the boundary control (3.13). The differential equation is

$$\frac{\partial}{\partial t}\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} + B(x)\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} + g(x)\, u(t) \quad . \tag{4.1}$$

Here $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is symmetric with eigenvalues ± 1, $B(x)$ is assumed a continuous 2×2 matrix on $[0, 1]$ and

$$g(x) = \begin{pmatrix} g^1(x) \\ g^2(x) \end{pmatrix}$$

is a vector in $L^2([0, 1]; E^2)$. The control u is scalar. The boundary conditions under consideration take the form

$$a_0 w^1(0, t) + b_0 w^2(0, t) = 0, \quad a_1 w^1(1, t) + b_1 w^2(1, t) = 0 \quad . \tag{4.2}$$

The requirement for existence and uniqueness of solutions both for t positive and for t negative, from the discussions before and after Theorem 3.1 of Section 3, here take the form

$$(a_1 - b_1)(a_0 + b_0) \neq 0 \quad ,$$

$$(a_1 + b_1)(a_0 - b_0) \neq 0 \quad ,$$

so that

$$\gamma \equiv (a_1 - b_1)(a_0 + b_0)/(a_1 + b_1)(a_0 - b_0) \tag{4.3}$$

is a well defined quantity.

An example arises very readily from the controlled wave equation

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} = \tilde{g}(x)\, u(t) \tag{4.4}$$

with boundary conditions

$$a_0 \frac{\partial w}{\partial t}(0,t) + b_0 \frac{\partial w}{\partial x}(0,t) = 0, \quad a_1 \frac{\partial w}{\partial t}(1,t) + b_1 \frac{\partial w}{\partial x}(1,t) = 0 \ . \tag{4.5}$$

With $w^1 \equiv \frac{\partial w}{\partial t}$, $w^2 \equiv \frac{\partial w}{\partial x}$, (4.4) gives

$$\frac{\partial w^1}{\partial t} = \frac{\partial w^2}{\partial x} + \tilde{g}(x)\, u(t) \tag{4.6}$$

and the familiar commutativity relation $\frac{\partial^2 w}{\partial t \partial x} = \frac{\partial^2 w}{\partial x \partial t} = \frac{\partial^2 w}{\partial x \partial t}$ becomes

$$\frac{\partial w^2}{\partial t} = \frac{\partial w^1}{\partial x} \ . \tag{4.7}$$

Equations (4.6), (4.7) combine to give (4.1) with

$$B(x) = 0, \quad g(x) = \begin{pmatrix} \tilde{g}(x) \\ 0 \end{pmatrix} \ .$$

Under the assumptions set forth above the operator

$$L\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} + B(x) \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \tag{4.8}$$

defined on the domain in $L^2([0,1]; E^2)$ consisting of two dimensional vector functions $\begin{pmatrix} w^1(x) \\ w^2(x) \end{pmatrix}$ with derivatives in $L^2([0,1]; E^2)$ and satisfying boundary conditions of the type (4.2) has a rather simple spectral

decomposition. The eigenvalues take the form (cf. (4.3))

$$\sigma_k = \tfrac{1}{2}\log\gamma + k\pi i + \varepsilon_k \equiv \alpha + k\pi i + \varepsilon_k , \tag{4.9}$$

$$-\infty < k < \infty ,$$

where

$$\varepsilon_k = \mathcal{O}\left(\frac{1}{|k|}\right), \quad |k| \to \infty .$$

The two dimensional eigenfunctions, $\phi_k(x)$, associated with these eigenvalues σ_k, are known to form a Riesz basis (linear isomorphic image of an orthonormal basis, see [66]) for the Hilbert space $L^2([0,1];E^2)$. Each $\begin{pmatrix} w^1 \\ w^2 \end{pmatrix}$ in this space has an expansion

$$\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = \sum_{k=-\infty}^{\infty} c_k \phi_k$$

and these are positive numbers d, D such that

$$d \left\| \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} \right\|^2_{L^2([0,1];E^2} \le \sum_{k=-\infty}^{\infty} |c_k|^2 \le D \left\| \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} \right\|^2_{L^2([0,1];E^2)} .$$

When $B(x) \equiv 0$ the ε_k are all zero and the σ_k and ϕ_k can be computed explicitly. In the general case we refer the reader to the very wide existing literature in eigenfunction expansions.

Since $g \in L^2([0,1];E^2)$ it has an expansion

$$g = \sum_{k=-\infty}^{\infty} g_k \phi_k .$$

It turns out that the appropriate controllability assumption is

$$g_k \neq 0, \quad -\infty < k < \infty \ ,$$

which amounts to saying that the "control distribution function" $g(x)$ has a non-zero component in the direction of each of the eigenfunctions $\phi_k(x)$.

We are also going to assume, for the sake of simplicity of presentation, that the σ_k are all distinct, which is not implied by (4.9) except in an asymptotic sense. This is not an essential restriction; the theory to be developed below can be extended without difficulty to the case where the operator L has multiple eigenvalues.

Let us pose the control problem here in terms of control from the zero initial state

$$\begin{pmatrix} w^1(\cdot, 0) \\ w^2(\cdot, 0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{4.10}$$

to a prescribed terminal state

$$\begin{pmatrix} w^1(\cdot, T) \\ w^2(\cdot, T) \end{pmatrix} = \begin{pmatrix} w_1^1 \\ w_1^2 \end{pmatrix} = \sum_{k=-\infty}^{\infty} c_k \phi_k \ , \tag{4.11}$$

the latter sum being the expansion of the given terminal state in terms of the eigenfunctions of the operator L . It is required to find a control $u \in L^2[0, T]$ steering a solution of (4.1), (4.2) from (4.10) to (4.11) during $[0, T]$. If we expand this solution in terms of the eigenfunctions ϕ_k of L :

$$\begin{pmatrix} w^1(\cdot, t) \\ w^2(\cdot, t) \end{pmatrix} = \sum_{k=-\infty}^{\infty} w_k(t)\, \phi_k \; ; \tag{4.12}$$

the coefficients $w_k(t)$ satisfy the scalar linear nonhomogeneous equations

$$\frac{dw_k}{dt} = \sigma_k w_k + g_k u(t), \qquad -\infty < k < \infty \; , \tag{4.13}$$

$$w_k(0) = 0, \qquad -\infty < k < \infty \; .$$

Consequently for any $T > 0$ we have

$$w_k(T) = \int_0^T e^{\sigma_k (T-t)} g_k u(t)dt, \qquad -\infty < k < \infty \; . \tag{4.14}$$

Comparing (4.11) and (4.14), our control objective is satisfied if and only if

$$c_k = \int_0^T e^{\sigma_k (2-t)} g_k u(t)dt, \qquad -\infty < k < \infty \; . \tag{4.15}$$

The set of equations (4.15) constitutes a moment problem for the unknown control function $u \in L^2[0, T]$. Such moment problems have a long and distinguished mathematical history. When $\alpha = 0$, $\varepsilon_k = 0$, $-\infty < k < \infty$, the functions $e^{\sigma_k t} = e^{k\pi i t}$ are the well studied functions of the Fourier series. It is familiar that these functions are linearly dependent in $L^2[0, T]$ if $T < 2$ (but, nevertheless, span the space). If $T > 2$ these functions fail to span $L^2[0, T]$ but are independent. Precisely for $T = 2$ we have both completeness and independence; indeed, the functions $\left(1/\sqrt{2}\right)e^{k\pi i t}$ form an orthonormal basis for $L^2[0, 2]$.

Classical results in harmonic analysis state that the "non-harmonic Fourier series", based on the functions $e^{\sigma_k t}$ (cf. (4.9)), behave in

much the same way as the usual Fourier series, based on the functions $e^{k\pi i t}$. The $e^{\sigma_k t}$ span $L^2[0,T]$, but are not independent, if $T > 2$. When $T = 2$ they form a Riesz basis for $L^2[0,2]$. What this means (previously stated for the eigenfunctions ϕ_k of L) is that there is a bounded and boundedly invertible linear transformation $F: L^2[0,2] \to L^2[0,2]$ such that

$$F(e^{k\pi i \cdot}) = e^{\sigma_k \cdot}, \qquad -\infty < k < \infty .$$

The operator $(F^*)^{-1}$ has some significance. If we set

$$(F^*)^{-1}(e^{k\pi i\cdot}) \equiv p_k, \quad -\infty < k < \infty$$

we have

$$(e^{\sigma_k \cdot}, p_\ell)_{L^2[0,2]} = (F(e^{k\pi i\cdot}), (F^*)^{-1}(e^{\ell\pi i\cdot}))_{L^2[0,2]} =$$

$$= (e^{k\pi i\cdot}, e^{\ell\pi i})_{L^2[0,2]} = \delta^k_\ell = \begin{cases} 1, & k = \ell \\ 0, & k = \ell . \end{cases}$$

This is summarized by saying that the p_k form a dual Riesz basis for the $e^{\sigma_k \cdot}$ in $L^2[0,2]$. This result, which has been known since the work of Paley and Wiener [73] in the 1930's, has been improved in various ways by, e.g., Levinson [56], Schwartz [99], Kadec [46], and many others. A more complete list of references is presented in [87]. A very easy proof, requiring little background, appears in [84]. For the most part the theory makes essential use of the Fourier transform and very technical estimates on the infinite product

$$\left(1 - \frac{z}{\sigma_0}\right) \prod_{k=1}^{\infty} \left[\left(1 - \frac{z}{\sigma_k}\right)\left(1 - \frac{z}{\sigma_{-k}}\right)\right]$$

(if any $\sigma_k = 0$ the corresponding term $1 - \frac{z}{\sigma_k}$ is replaced by z) and the associated products obtained by eliminating one factor at a time. This approach requires $\alpha = 0$ but does not at all require that the ε_k in (4.9) be square summable. It is enough to have $|\varepsilon_k| \leq \rho < \frac{\pi}{4}$ for all sufficiently large k. The result can trivially be extended to $\alpha \neq 0$ just by multiplying each of the functions $e^{(k\pi i+\varepsilon_k)t}$ by $e^{\alpha t}$ and each of the functions p_ℓ dual to $e^{(k\pi i+\varepsilon_k)t}$ by $e^{-\bar{\alpha} t}$.

With the above background we can immediately provide a formal solution for the moment problem (4.15) when $T = 2$, i.e., for

$$c_k = \int_0^2 e^{\sigma_k(2-t)} g_k\, u(t)dt, \qquad -\infty < k < \infty . \tag{4.16}$$

For convenience we redefine the $p_\ell(t)$ by $p_\ell(2-t)$) so that

$$\int_0^2 e^{\sigma_k(2-t)} \overline{p_\ell(t)}\, dt = \delta_\ell^k . \tag{4.17}$$

Then

$$u(t) = \sum_{\ell=-\infty}^{\infty} \left(\frac{c_\ell}{g_\ell}\right) \overline{p_\ell(t)} \tag{4.18}$$

solves (4.16) in a formal sense. We have a bona fide solution $u \in L^2[0,2]$ just in case

$$\sum_{\ell=-\infty}^{\infty} \left|\frac{c_\ell}{g_\ell}\right|^2 < \infty . \tag{4.19}$$

(Because (4.18) is the image under $(F^*)^{-1}$ of $\sum_{\ell=-\infty}^{\infty} (\frac{c_\ell}{g_\ell}) e^{\ell \zeta i t})$. The fact that the $e^{\sigma_k t}$ are linearly dependent in $L^2[0, T]$ for $T < 2$ shows that (4.15) cannot be solved in general, even formally, if $T < 2$. The fact that the e^{σ_k} are linearly independent, but fail to span $L^2[0, T]$, when $T \geq 2$, implies that (4.15) can be solved, but not uniquely, when $T \geq 2$. We see then that our control problem (4.1), (4.2), (4.10), (4.11) has a unique solution in $L^2[0, 2]$ just in case $T = 2$ and (4.19) holds. Referring to the general abstract framework of Section 2, with $Y = L^2[0, 2]$, $Z = L^2([0, 2]; E^2)$, the operator C is defined by

$$Cu = \sum_{k=-\infty}^{\infty} \left(\int_0^2 e^{\sigma_k(2-t)} g_k u(t)dt\right)\phi_k \tag{4.20}$$

and maps $L^2[0, 2]$ into a dense subspace $G \subseteq L^2([0, 2]; E^2)$ consisting of sequences $\sum_{k=-\infty}^{\infty} c_k \phi_k$ *for which* (4.19) is true. Taking X also to be $L^2([0, 2]; E^2)$, if we choose $S = I$, $\{X, Y, Z, S, C\}$ is approximately controllable. On the other hand if we let S be defined by $S \phi_k = g_k \phi_k$, $-\infty < k < \infty$, then $\{X, Y, Z, S, C\}$ is (exactly) controllable.

If we consider instead of (4.1), (4.2) the boundary control system

$$\frac{\partial}{\partial t}\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} + B(x)\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} , \tag{4.21}$$

$$a_0 w^1(0, t) + b_0 w^2(0, t) = 0 , \tag{4.22}$$

$$a_1 w^1(1, t) + b_1 w^2(1, t) = u(t) \tag{4.23}$$

the results of Theorem 3.2 show the system to be uniquely (since there

are in this case no intervals I_k on which arbitrary data need be specified) controllable just in case $T = 2$. That this time coincides with that developed above is, of course, hardly a mere coincidence. One can show (see [90] for details) that the control problem (4.10), (4.11), posed for (4.21), (4.22), (4.23) is equivalent to a moment problem

$$c_k = \int_0^2 e^{\sigma_k(2-t)} \hat{g}_k \, u(t)dt, \qquad -\infty < k < \infty \tag{4.24}$$

with

$$\hat{g}_k = \begin{cases} a_1^{-1} \overline{\phi_k^2(1)}, & a_1 \neq 0 \ , \\ b_1^{-1} \overline{\phi_k^1(1)}, & b_1 \neq 0 \ . \end{cases}$$

Analysis of the differential equation

$$\sigma_k \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} + B(x) \begin{pmatrix} w^1 \\ w^2 \end{pmatrix}$$

satisfied by ϕ_k enables one to see that there are positive numbers γ_0 and γ_1 such that

$$\gamma_0 \leq |\hat{g}_k| \leq \gamma_1, \qquad -\infty < k < \infty \ .$$

In this case, therefore, the range space $\hat{G}$ of

$$\hat{C} u = \sum_{k=-\infty}^{\infty} \left(\int_0^2 e^{\sigma_k(2-t)} \hat{g}_k \, u(t)dt \right) \phi_k \tag{4.25}$$

is precisely $L^2([0,2]; E^2)$, just as we would have from Theorem 3.2, and $\{X, Y, Z, \hat{S}, \hat{C}\}$ is exactly controllable with $S = I$. In fact, this

route, Theorem 3.2 followed by the observation that solution of (4.24) is equivalent to solution of the control problem, provides another way of establishing the existence of biorthogonal functions p_ℓ satisfying (4.17); the functions p_ℓ are precisely the controls steering the zero initial state to the final state

$$\begin{pmatrix} w^1(\cdot, 2) \\ w^2(\cdot, 2) \end{pmatrix} = \hat{g}_\ell \, \phi_\ell = \hat{g}_\ell \sum_{k=-\infty}^{\infty} \delta_\ell^k \, \phi_k \, .$$

Thus the relationship between control theory of hyperbolic systems and harmonic analysis is a two-way affair. One can begin with results in either field and infer corresponding results in the other. In Section 8 this program will be carried out in connection with a more complex control situation.

Not all control problems for the scalar wave equation (4.4) can be treated in the context of (4.1). If we take (4.4) with boundary conditions $w(0,t) = w(1,t) = 0$, for example, the resulting moment problems involve only $e^{k\pi i t}$, $k = \pm 1, \pm 2, \ldots$; the function 1 is missing. Control is not then unique even for $T = 2$. If we take $a_0 = a_1 = 0$, $b_0 = b_1 = 1$ in (4.5) we have a multiple zero eigenvalue and the moment problems involve the functions $1, t, e^{k\pi i t}$, $k = \pm 1, \pm 2, \ldots$. These functions are not independent in $L^2[0,2]$ and control is possible only if $T > 2$. (The eigenstate $w = 1$, $\frac{\partial w}{\partial t} = 0$ is suppressed in passing from (4.4) to (4.1) via $w^1 = \frac{\partial w}{\partial t}$, $w^2 = \frac{\partial w}{\partial x}$.) These and other problems of the same sort are discussed in [87]. There are a large number of open questions

associated with more complex systems wherein the eigenvalues of the generator of the semigroup occur as several sequences of the type (4.9) but with various asymptotic intervals between the members of the individual sequences – or the eigenvalues may occur in clusters with the clusters themselves having an asymptotic separation. Some rather nice theoretical results have been obtained along these lines by Ulrich [107].

We have noted earlier, in Theorem 2.9, that if the finite dimensional linear control system (2.1) is controllable then not only is it stabilizable but the eigenvalues of the closed loop matrix can be selected at will by appropriate choice of the feedback matrix K. We are going to complete this section by describing a comparable result for the system (4.1), (4.2) (or, equivalently, for (4.21)-(4.23)). To motivate the procedures which we will use, we briefly review the proof of the "spectral assignment" property claimed in Theorem 2.9. We do this only for the case of a scalar control u; the case $u \in E^m$ is fully treated in [6], [111].

Thus we consider the system

$$\dot{x} = Ax + bu, \quad x \in E^n, \quad u \in E^1 \tag{4.26}$$

and the closed loop systems

$$\dot{x} = (A + bk^*)x \tag{4.27}$$

obtainable from (4.26) with use of linear feedback relations

$$u = k^* x, \quad k \in E^n . \tag{4.28}$$

We assume (4.26) to be controllable. Then

$$C = (A^{n-1}b,\ A^{n-2}b,\ \ldots,\ Ab,\ b)$$

is a nonsingular $n \times n$ matrix. It bears a definite relationship to the operator $\mathcal{C}$ defined just prior to Definition 2.4, representing, as it does, a sort of "control to state" map. If one considers distribution valued controls with $\{0\}$ as support,

$$u = \tilde{\zeta}^1 \delta_0^{n-1} + \tilde{\zeta}^2 \delta_0^{n-2} + \ldots + \tilde{\zeta}^n \delta_0 ,$$

(δ_0 is the Dirac "delta function" with $\{0\}$ as support and δ_0^k is its k-th derivative in the sense of the theory of distributions) and assumes

$$x(0-) = 0$$

one may readily verify that

$$x(0+) = A^{n-1} b\tilde{\zeta}^1 + A^{n-2} b\tilde{\zeta}^2 + \ldots + b\tilde{\zeta}^n = C\begin{pmatrix} \tilde{\zeta}^1 \\ \tilde{\zeta}^2 \\ \vdots \\ \tilde{\zeta}^n \end{pmatrix} .$$

Since C is nonsingular it can be used to transform the dynamical system (4.26) in E^n into another system in the space $\tilde{E}^n$ consisting of vectors

$$\tilde{\zeta} = \begin{pmatrix} \tilde{\zeta}^1 \\ \tilde{\zeta}^2 \\ \vdots \\ \tilde{\zeta}^n \end{pmatrix} .$$

Indeed, putting

$$x = C\tilde{\zeta}$$

(4.26) becomes

$$\dot{\tilde{\zeta}} = C^{-1} AC\tilde{\zeta} + C^{-1}bu \equiv \tilde{A}\,\tilde{\zeta} + \tilde{b}u \ . \qquad (4.29)$$

Since the last $n-1$ columns of AC agree with the first $n-1$ columns of C and since b is the last column of C, $\tilde{A}$ and $\tilde{b}$ must take the form

$$\tilde{A} = \begin{pmatrix} -a_1 & 1 & 0 & & 0 & 0 \\ -a_2 & 0 & 0 & & 0 & 0 \\ \vdots & & & & & \\ -a_{n-1} & 0 & 0 & & 0 & 1 \\ -a_n & 0 & 0 & & 0 & 0 \end{pmatrix}, \qquad \tilde{b} = e_n \equiv \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \qquad (4.30)$$

where $a_1, a_2, \ldots, a_{n-1}, a_n$ are such that

$$A^n b + a_1 A^{n-1} b + \ldots + a_{n-1} Ab + a_n b = 0$$

(and hence are by the Cayley-Hamilton theorem, the coefficients of the characteristic polynomial of $\tilde{A}$ (and A)). The equation (4.30) constitutes the <u>control normal form</u> for the system (4.26).

The next step is not particularly easy to motivate <u>a priori</u>. We let

$$\Phi = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & \cdots & 0 \\ a_2 & a_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-2} & a_{n-3} & a_{n-4} & \cdots & 0 \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & 1 \end{pmatrix}. \tag{4.31}$$

Evidently Φ is nonsingular and setting

$$\tilde{\zeta} = \Phi \zeta$$

one may verify with a little calculation that (4.29) is transformed to

$$\dot{\zeta} = \hat{A} \zeta + e_n u \ , \tag{4.32}$$

e_n, as in (4.30), remaining invariant and

$$\hat{A} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{pmatrix} \tag{4.33}$$

The form (4.32) is known as the <u>control canonical form</u> for (4.26).

In (4.32) the effect of feedback

$$u = \hat{k}^* \zeta \ \ (= \hat{k}^* \Phi^{-1} C^{-1} x) \tag{4.34}$$

is immediately visible. In the closed loop system

$$\dot{\zeta} = (\hat{A} + e_n \hat{k}^*)\zeta$$

the matrix has the form

$$\hat{A} + e_n \hat{k}^* = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_n + \hat{k}_1 & -a_{n-1} + \hat{k}_2 & \cdots & -a_1 + \hat{k}_n \end{pmatrix} .$$

The characteristic polynomial of $\hat{A} + e_n \hat{k}^*$ is

$$p(\lambda) = \lambda^n + (a_1 - \hat{k}_n)\lambda^{n-1} + \ldots + (a_{n-1} - \hat{k}_2)\lambda + (a_n - \hat{k}_1)$$

and, since its coefficients can clearly be selected at will by appropriate choice of $\hat{k}$ in (4.34) (and hence of k in (4.28)) its zeroes can also be selected at will, thereby providing a proof for *Theorem 2.9 in the* case of a scalar control.

To give some meaning to Φ, shown in (4.31), let us note that, with

$$\Sigma = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & ⋰ & \vdots & \vdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

we have $\Sigma = \Sigma^{-1}$ and (with $\overline{}$ denoting complex conjugate)

$$\hat{A}^* = \Sigma \tilde{A}^T \Sigma = \overline{\Sigma \tilde{A} \Sigma} . \qquad (4.35)$$

Suppose $\tilde{A}$ has eigenvectors $v_1, v_2, \ldots, v_n$ and eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$. Then $\tilde{A}^*$ has as its eigenvectors the dual vectors $w_1, w_2, \ldots, w_n$ for which $(v_i, w_j)_{E^n} = \delta^i_j$ and $\tilde{A}^* w_j = \bar{\mu}_j w_j$. Then

$$\overline{\Sigma \tilde{A} \Sigma} (\Sigma \bar{v}_j) = \overline{\Sigma \tilde{A} v_j} = \bar{\mu}_j \Sigma \bar{v}_j \; . \tag{4.36}$$

Combining (4.35) and (4.36) we see that

$$\hat{A}^* \Sigma \bar{v}_j = \bar{\mu}_j \Sigma \bar{v}_j$$

and we conclude that $\Sigma \bar{v}_j$ is an eigenvector of $\hat{A}^*$ corresponding to the eigenvalue $\bar{\mu}_j$. But, since

$$\hat{A}^* = \Phi^* \tilde{A}^* (\Phi^*)^{-1}$$

the eigenvectors of $\hat{A}^*$ have the form $\Phi^* w_j$, where the w_j are the eigenvectors of $\tilde{A}^*$. Hence

$$\Phi^* w_j = \beta_j \Sigma \bar{v}_j \tag{4.37}$$

for some $\beta_j \neq 0$, i.e., Φ^* takes the vector w_j of the dual basis $(w_1, w_2, \ldots, w_n)$ into a scalar multiple of the "order reversed" complex conjugate of the corresponding element v_j of the original basis $(v_1, v_2, \ldots, v_n)$. *This will be significant in a later comparison.*

Our next objective is to develop, for (4.1), (4.2), a control canonical form analogous to (4.32), (4.33) and playing a comparable role with respect to spectral determination via feedback. Let $w_k(t)$, $-\infty < k < \infty$, denote a solution sequence of (4.13), i.e.,

$$\frac{dw_k}{dt} = \sigma_k w_k + g_k u(t)$$

lying in G (cf. (4.20) ff) for all t – which is true if it lies in G for any t. We use the control-to-state map C exhibited in (4.20) to define a new variable $\tilde{\zeta} = \tilde{\zeta}(t, \tau)$:

$$w_k(t) = (C\,\tilde{\zeta}(t, \cdot))_k(t) = \int_0^2 e^{\sigma_k(2-\tau)} g_k \tilde{\zeta}(t, \tau) d\tau \ . \tag{4.38}$$

The inverse map, from (4.18), is

$$\tilde{\zeta}(t, \tau) = \sum_{k=-\infty}^{\infty} \frac{w_k(t)}{g_k} \overline{p_k(\tau)} \ . \tag{4.39}$$

Our task now will be to see how the dynamical equations (4.13) for the $w_k(t)$ are transformed into dynamical equations for $\tilde{\zeta}(t, \tau)$ under the above transformation. Before doing so, however, it will be useful to consider a certain relationship which is satisfied by the exponential functions $e^{\sigma_k t}$. If we had $\sigma_k = \alpha + k\pi i$ we would have

$$e^{\sigma_k 2} = e^{(\alpha + k\pi i)2} = e^{2\alpha} (= e^{\alpha} e^{\sigma_k 0})$$

and every linear combination

$$r(t) = \sum_k r_k e^{\sigma_k t}$$

would likewise satisfy

$$r(2) = e^{2\alpha} r(0) \ .$$

Let us form the function

$$p(t) = \sum_{k=-\infty}^{\infty} (e^{\bar{\sigma}_k 2} - e^{2\bar{\alpha}})\, p_j(t) \ . \qquad (4.41)$$

Since $e^{\sigma_k 2} = e^{(\alpha + k\pi i + \mathcal{O}(\frac{1}{k}))2} = e^{2\alpha} e^{\mathcal{O}(\frac{1}{k})} = e^{2\alpha}(1 + \mathcal{O}(\frac{1}{k})) = e^{2\alpha} + \mathcal{O}(\frac{1}{k})$ the series (4.41) converges and $p \in L^2[0,2]$. Since $(e^{\sigma_k(2-\cdot)}, p_j)_{L^2[0,2]} = \delta_j^k$,

$$\int_0^2 e^{\sigma_k(2-\tau)}\, \overline{p(\tau)}\, d\tau = \int_0^2 (e^{\sigma_k 2} - e^{2\alpha})\, e^{\sigma_k(2-\tau)}\, p_k(\tau) d\tau = e^{\sigma_k 2} - e^{2\alpha} \ .$$

Each linear combination (4.40) with $\sigma_k = \alpha + k\pi i + \mathcal{O}(\frac{1}{k})$ consequently satisfies

$$r(2) = e^{2\alpha} r(0) + \int_0^2 \overline{p(2-\tau)}\, r(\tau) d\tau \ .$$

To find the dynamical equation satisfied by $\tilde{\zeta}(t,\tau)$ we differentiate (4.39) formally and then use (4.13) and (4.38):

$$\frac{\partial \tilde{\zeta}(t,\tau)}{\partial t} = \sum_{k=-\infty}^{\infty} \frac{\frac{dw_k}{dt}}{gk}\, \overline{p_k(\tau)} =$$

$$= \sum_{k=-\infty}^{\infty} \frac{\sigma_k w_k(t) + g_k u(t)}{g_k}\, \overline{p_k(\tau)} =$$

$$= \sum_{k=-\infty}^{\infty} \sigma_k \int_0^2 e^{\sigma_k(2-\tau)} \tilde{\zeta}(t,\tau) d\tau\, \overline{p_k(\tau)} + u(t) \sum_{k=-\infty}^{\infty} \overline{p_k(\tau)}$$

= (integrating by parts)

$$= \sum_{k=-\infty}^{\infty} \left(\int_0^2 e^{\sigma_k(2-\tau)} \frac{\partial \tilde{\zeta}(t,\tau)}{\partial \tau} d\tau\right) \overline{p_k(\tau)}$$

$$+ \sum_{k=-\infty}^{\infty} (e^{2\sigma_k} \tilde{\zeta}(t,0) - \tilde{\zeta}(t,2) + u(t))\, \overline{p_k(\tau)} \qquad (4.42)$$

Adding and subtracting $e^{2\alpha}\tilde{\zeta}(t,0)$, the second sum above becomes

$$\sum_{k=-\infty}^{\infty}(e^{2\sigma_k}-e^{2\alpha})\tilde{\zeta}(t,0)\overline{p_k(\tau)}$$

$$+\sum_{k=-\infty}^{\infty}(e^{2\alpha}\tilde{\zeta}(t,0)-\tilde{\zeta}(t,2)+u(t))\overline{p_k(\tau)} \quad . \tag{4.43}$$

The second sum here is just a multiple of $\sum_{k=-\infty}^{\infty}\overline{p_k(\tau)}$ which can be shown to be the distribution $\delta(2-\tau)$. We eliminate it by requiring

$$\tilde{\zeta}(t,2)=e^{2\alpha}\tilde{\zeta}(t,0)+u(t) \quad . \tag{4.44}$$

From (4.41) the first term in (4.43) is seen to be just $\overline{p(\tau)}\ \tilde{\zeta}(t,0)$. So now (4.42) gives

$$\frac{\partial\tilde{\zeta}(t,\tau)}{\partial t}=\sum_{k=-\infty}^{\infty}\frac{\left(\int_0^2 e^{\sigma_k(2-\tau)}g_k\frac{\partial\tilde{\zeta}(t,\tau)}{\partial\tau}d\tau\right)}{g_k}\overline{p_k(\tau)}$$

$$+\overline{p(\tau)}\ \tilde{\zeta}(t,0) \ .$$

But the sum above is just (see (4.38) and (4.39)) $C^{-1}C(\frac{\partial\tilde{\zeta}}{\partial\tau})=\frac{\partial\tilde{\zeta}}{\partial\tau}$. So we have

$$\frac{\partial\tilde{\zeta}}{\partial t}=\frac{\partial\tilde{\zeta}}{\partial\tau}+\overline{p(\tau)}\ \tilde{\zeta}(t,0) \quad . \tag{4.45}$$

The equations (4.45) and (4.44) constitute the control normal form for (4.13) (hence (4.1), (4.2) and should be compared with (4.29), (4.30).

The above derivation is entirely formal. In [95] it is justified by making use of the observation normal form for the dual observed system.

To pass to the control canonical form we employ a transformation

$\tilde{\zeta} = \Phi\zeta$ of "convolution type" which may be regarded as a distributed analog of (4.31):

$$\tilde{\zeta}(t,\tau) = (\Phi\zeta(t,\cdot))(\tau) = \zeta(t,\tau) - \int_0^\tau \overline{p(\tau-\sigma)}\,\zeta(t,\sigma)d\sigma \quad . \tag{4.46}$$

Substituting this expression in (4.44) we have

$$\zeta(t,2) - \int_0^2 \overline{p(2-\sigma)}\,\zeta(t,\sigma)\,d\sigma = e^{2\alpha}\,\zeta(t,0) + u(t)$$

whence, renaming the variable of integration,

$$\zeta(t,2) = e^{2\alpha}\,\zeta(t,0) + \int_0^2 \overline{p(2-\tau)}\,\zeta(t,\tau)d\tau + u(t) \quad . \tag{4.47}$$

To see how (4.45) transforms we observe that

$$\begin{aligned}
0 &= \frac{\partial\tilde{\zeta}(t,\tau)}{\partial t} - \frac{\partial\tilde{\zeta}(t,\tau)}{\partial\tau} - \overline{p(\tau)}\,\tilde{\zeta}(t,0) \\
&= \frac{\partial}{\partial t}(\zeta(t,\tau) - \int_0^\tau \overline{p(\tau-\sigma)}\,\zeta(t,\sigma)d\sigma) \\
&\quad - \frac{\partial}{\partial\tau}(\zeta(t,\tau) - \int_0^\tau \overline{p(\tau-\sigma)}\,\zeta(t,\sigma)\,d\sigma) - \overline{p(\tau)}\,\zeta(t,0) \\
&= \frac{\partial\zeta(t,\tau)}{\partial t} - \int_0^\tau \overline{p(\tau-\sigma)}\,\frac{\partial\zeta(t,\sigma)}{\partial t}d\sigma - \frac{\partial\zeta(t,\tau)}{\partial\tau} + \overline{p(0)}\,\zeta(t,\tau) \\
&\quad + \int_0^\tau \frac{\partial\overline{p(\tau-\sigma)}}{\partial\tau}\,\zeta(t,\sigma)d\sigma - \overline{p(\tau)}\,\zeta(t,0) \quad .
\end{aligned}$$

Noting that $\frac{\partial p(\tau-\sigma)}{\partial\tau} = -\frac{\partial\overline{p(\tau-\sigma)}}{\partial\sigma}$ and integrating by parts in the second integral above we obtain

$$\frac{\partial\zeta(t,\tau)}{\partial t} - \frac{\partial\zeta(t,\tau)}{\partial\tau} = \int_0^\tau \overline{p(\tau-\sigma)}\,\left(\frac{\partial\zeta(t,\sigma)}{\partial t} - \frac{\partial\zeta(t,\sigma)}{\partial\sigma}\right)d\sigma$$

from which we conclude that

$$\frac{\partial \zeta(t,\tau)}{\partial t} - \frac{\partial \zeta(t,\tau)}{\partial \tau} = 0 \ . \tag{4.48}$$

Now (4.48) shows that ζ is constant on lines $t + \tau = c$ (constant). We may therefore write ζ as $\zeta(t + \tau)$ and (4.47) becomes

$$\zeta(t+2) = e^{2\alpha}\,\zeta(t) + \int_0^2 \overline{p(2-\tau)}\,\zeta(t+\tau)d\tau + u(t) \ . \tag{4.49}$$

This is a neutral delay equation which constitutes the control canonical form for the system (4.13) (hence (4.1), (4.2)). The distributed part of the delay vanishes when $B(x) \equiv 0$ in (4.1) since $e^{\sigma_k 2} = e^{2\alpha}$ in that case for all k and p (cf. (4.41)) is thus identically equal to zero. This canonical form should be compared with (4.32), (4.33) of which it is a distributed analog.

Just as the control canonical form for finite dimensional systems enables one to see that eigenvalues of the closed loop system can be placed at will by appropriate choice of the feedback vector k^*, the canonical form just obtained can be used in a similar way with respect to the system (4.13) (or, equivalently, (4.1), (4.2)).

Let us suppose that we determine the control $u(t)$ by means of the linear feedback law

$$u(t) = \left(\begin{pmatrix} w^1(\cdot, t) \\ w^2(\cdot, t) \end{pmatrix}, k \right)_{L^2([0,1];E^2)} , \tag{4.50}$$

where

$$k = \begin{pmatrix} k^1 \\ k^2 \end{pmatrix} \in L^2([0,1];E^2), \quad k = \sum_{j=-\infty}^{\infty} k_j \phi_j \ .$$

The closed loop system thus obtained involves addition of a "dyadic operator" to the generator $L\begin{pmatrix} w^1 \\ w^2 \end{pmatrix}$ (cf. (4.8)):

$$\frac{\partial}{\partial t}\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = L_k\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} + B(x)\begin{pmatrix} w^1 \\ w^2 \end{pmatrix} + g(x)\left(\begin{pmatrix} w^1 \\ w^2 \end{pmatrix}, k\right). \quad (4.51)$$

Our next objective is to indicate how this closed loop system transforms under the action of the mappings C and Φ above (cf. (4.38), (4.46)).

In terms of the variables present in the system (4.13) it is clear that the feedback law (4.50) must take the form

$$u(t) = \sum_{j=-\infty}^{\infty} \overline{k_j}\; w_j(t)$$

and, by virtue of (4.38), in terms of the variable $\tilde{\zeta}$ of the control normal form we have

$$u(t) = \sum_{j=-\infty}^{\infty} \overline{k_j}\; g_j \int_0^2 e^{\sigma_j(2-\tau)}\, \tilde{\zeta}(t,\tau)\, d\tau \ .$$

From (4.46) we then have

$$u(t) = \sum_{j=-\infty}^{\infty} \overline{k_j}\; g_j \int_0^2 e^{\sigma_j(2-\tau)}\, \Phi\zeta(t,\tau)\, d\tau$$

$$= \sum_{j=-\infty}^{\infty} \overline{k_j}\; g_j \int_0^2 \overline{(\Phi^* e^{\bar{\sigma}_j(2-\cdot)})(\tau)}\;\; \zeta(t,\tau)\, d\tau \ .$$

The following proposition is perhaps the most important and most difficult of the whole theory. We refer the reader to [95] or [96] for (two different) proofs and more details.

Proposition 4.1. $(\Phi^* e^{\bar{\sigma}_j(2-\cdot)})(\tau) = \beta_j \overline{p_j(2-\tau)}$ where p_j is the biothogonal function defined by (4.17) and $0 < b \leq \beta_j \leq B$, $-\infty < j < \infty$.

The result stated in this proposition is precisely the analog of the formula (4.37) developed earlier for the finite dimensional case. In addition this result shows that the map Φ is, except for a trivial change of independent variable, the map used by N. Bari in her treatment of Riesz bases (see discussion and reference in [66]).

We finally have, then,

$$u(t) = \sum_{j=-\infty}^{\infty} \bar{k}_j g_j \int_0^2 \overline{\beta_j p_j(2-\tau)}\ \zeta(t,\tau)d\tau$$

$$= \int_0^2 \overline{k_g(2-\tau)}\ \zeta(t,\tau)d\tau \tag{4.52}$$

where

$$k_g(t) = \sum_{j=-\infty}^{\infty} k_j \overline{g_j} \beta_j p_j(2-\tau) \quad . \tag{4.53}$$

Consequently, in terms of the canonical variable ζ, the closed loop system (4.51) becomes (cf. (4.49))

$$\zeta(t+2) = e^{2\alpha} \zeta(t) + \int_0^2 (\overline{p(2-\tau)} + \overline{k_g(2-\tau)})\ \zeta(t+\tau)d\tau \quad . \tag{4.54}$$

We see, therefore, that the control canonical form has the property of being invariant under feedback. A feedback relation (4.50) just modifies the kernel $\overline{p(2-\tau)}$ in the distributed delay term. It is this invariance property which enables us to study how the spectrum of the closed loop

system (4.51) can be modified when u obeys a feedback law (4.50).

The main result is

Theorem 4.2. Let distinct complex numbers ρ_j, $-\infty < j < \infty$, be selected with the property

$$\sum_{j=-\infty}^{\infty} \left| \frac{\sigma_j - \rho_j}{g_j} \right|^2 < \infty \quad . \tag{4.55}$$

Then there is a feedback vector function $k \in L_2^2[0,1]$ for which the closed loop system (4.51) has precisely the eigenvalues ρ_j.

Sketch of Proof. Consider a system (cf. (4.13))

$$\frac{d\hat{w}_j}{dt} = \rho_j \hat{w}_j, \quad -\infty < j < \infty$$

having the desired ρ_j as eigenvalues. Reducing this system to canonical form one obtains (cf. (4.49))

$$\hat{\zeta}(t+2) = e^{2\alpha} \hat{\zeta}(t) + \int_0^2 \overline{q(2-\tau)} \, \hat{\zeta}(t+\tau) d\tau$$

where (cf. (4.41), (4.17))

$$q(\tau) = \sum_{j=-\infty}^{\infty} (e^{\bar{\rho}_j 2} - e^{2\bar{\alpha}}) q_j(\tau) \quad ,$$

$$\int_0^2 e^{\rho_j (2-\tau)} \overline{q_k(\tau)} \, d\tau = \delta_j^k \quad .$$

To realize the ρ_j as eigenvalues of the closed loop system (4.51) it is necessary and sufficient that

$$\overline{p(2-\tau)} + k_g \overline{(2-\tau)} = \overline{q(2-\tau)}$$

Referring to (4. 52) we see that the question is whether or not there is a sequence $\{k_j\}$ with $\sum_{j=-\infty}^{\infty} |k_j|^2 < \infty$ such that

$$\sum_{j=-\infty}^{\infty} \overline{k_j}\, g_j\, \overline{\beta_j}\, p_j\overline{(2-\tau)} = \overline{q(2-\tau)} - \overline{p(2-\tau)} \ .$$

With a rather simple calculation, which just involves the estimation of the expansion coefficients of $\overline{q(2-\tau)}$ with respect to the functions $p_j\overline{(2-\tau)}$ it is shown in [95] that this can be done if the inequality (4. 55) is satisfied. The calculation does not show that (4. 55) is necessary but it does indicate that no significant improvement is likely.

We have noted that in the case of boundary control ((4. 21), (4. 22), (4. 23)) the $\hat{g}_j$ (cf. (4. 24)) are bounded and bounded away from zero. It can be shown in this case that, by utilization of feedback relationships of the form (4. 50), one can realize values ρ_j, $-\infty < j < \infty$, for the closed loop system provided

$$\sum_{j=-\infty}^{\infty} |\sigma_j - \rho_j|^2 < \infty \ . \tag{4. 56}$$

In both cases, (4. 55) and (4. 56), any finite subcollection of the points of the spectrum can be moved at will. Indeed, the whole spectrum can be moved "at will" in a sense. If one selects a sequence of disc neighborhoods N_j of radius r_j centered at σ_j with $\sum_{j=-\infty}^{\infty} \left|\frac{r_j}{g_j}\right|^2 < \infty$ ($\sum_{j=-\infty}^{\infty} |r_j|^2 < \infty$ in the boundary control case) then the ρ_j can be placed at will in the neighborhoods N_j .

If, in addition to having boundary control, as in (4.21), (4.22), (4.23), one likewise permits boundary feedback, e.g.

$$u(t) = \alpha_1 w^1(1,t) + \beta_1 w^2(1,t) + \left(\binom{w^1}{w^2}, k\right)_{L^2([0,1];E^2)} .$$

We then obtain, as the closed loop system, (4.21),

$$(a_1 - \alpha_1) w^1(1,t) + (b_1 - \beta_1) w^2(1,t) = \left(\binom{w^1}{w^2}, k\right)_{L^2([0,1];E^2)} \tag{4.57}$$

In this case one can change the "base point" α (cf. (4.9)) of the spectrum to any other desired base point β and (4.56) is replaced by

$$\sum_{j=-\infty}^{\infty} |(\sigma_j - \rho_j) - (\alpha - \beta)|^2 < \infty .$$

Thus the asymptotic vertical line along which the spectrum is located can be moved and then square summable changes in the σ_j can be carried out as well.

One of the outstanding problems remaining in this area is to extend the notion of control canonical form from the very special class of distributed systems considered here to, first of all, higher dimensional hyperbolic systems and parabolic systems and, secondly, but even more fundamentally, to a general abstract framework which will include a large class of linear distributed parameter systems. While very little can be said now about such a project it is safe to predict that the "control to state map" C and the map Φ, which we have seen relates a Riesz basis to its dual basis, will remain important in the theory.

5. BOUNDARY CONTROLLABILITY THEORY FOR HIGHER DIMENSIONAL HYPERBOLIC SYSTEMS

In contrast with the relatively complete theory which we have seen to exist for hyperbolic equations involving only a single space variable x, confined to an interval $0 \leq x \leq 1$, the control and observation theory for processes taking place in multi-dimensional regions is quite primitive. This is due to the fact that the characteristic surfaces arising in such problems are nowhere near as constructively useful as in the one dimensional case. Nevertheless such theory as does exist does not lack interest, as we shall point out.

Virtually the only multidimensional hyperbolic control process which has received any attention is the scalar wave equation, which we shall write here in the form

$$\rho(x) \frac{\partial^2 w}{\partial t^2} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(\alpha_j^i(x) \frac{\partial w}{\partial x^j} \right) = 0 \ , \tag{5.1}$$

where $\rho(x)$, $\alpha_{ij}(x)$ are real valued functions of class C^2 in the closure of Ω, a bounded open connected set in R^n with piecewise smooth (plus other conditions as required for existence, uniqueness, etc.) boundary Γ. It is assumed that

$$\rho(x) \geq \rho_0 > 0, \quad x \in \Omega \ , \tag{5.2}$$

$$(w, A(x)w)_{E^n} \geq \alpha_0 \|w\|^2_{E^n}, \quad \alpha_0 > 0 \ , \tag{5.3}$$

where $A(x)$ is the $n \times n$ symmetric matrix with entries $\alpha_j^i(x)$. The condition (5.3) is, of course, the familiar "uniform ellipticity" condition.

We denote by $\nu(x)$ the (almost everywhere defined and unique) unit exterior normal to $\Gamma = \partial\Omega$ at $x \in \Gamma$. We assume Γ divided into two parts

$$\Gamma = \Gamma_0 \cup \Gamma_1, \quad \Gamma_0 \cap \Gamma_1 = \emptyset , \tag{5.4}$$

with Γ_1 non-empty and relatively open in Γ. Also, unless explicitly stated to the contrary, we will assume Γ_0 to have a non-empty interior as well.

The idea now is that control will be exercised on Γ_1 while only natural, passive constraints, in the form of boundary conditions, will act on Γ_0. This still leaves a number of possibilities for different types of control application and different types of constraints. To avoid inessential complication in this expository treatment we shall consider only solutions $w = w(x, t)$ of (5.1) for which

$$w(x, t) \equiv 0, \quad x \in \Gamma_0 \tag{5.5}$$

$$(\nu(x), \ A(x) \frac{\partial w}{\partial x}(x, t))_{E^n} \equiv u(x, t), \quad x \in \Gamma_1 \tag{5.6}$$

u being a scalar control function restricted to lie in $L^2(\Gamma_1 \times [0, T])$ for any fixed $T > 0$. Thus, again, we are concerned with a "boundary control".

It is by now an almost classical result that if one gives initial data

$$w(x, 0) = w_0(x), \ x \in \Omega, \ w_0 \in H^m(\Omega) \tag{5.7}$$

$$\frac{\partial w}{\partial t}(x, 0) = v_0(x), \ x \in \Omega, \ v_0 \in H^{m-1}(\Omega) , \tag{5.8}$$

(where $H^m(\Omega)$ is the usual Sobolev space [1]) and if appropriate consistency conditions relating w_0, v_0 to the boundary conditions (5.5), (5.6) are met, then (5.1), (5.5), (5.6), (5.7), (5.8) has a unique generalized solution $w \in \mathcal{C}([0, \infty); H^m(\Omega) \oplus H^{m-1}(\Omega))$ in the case where $u(x, t) \equiv 0$. When a simple matter to relate regularity of u to regularity of the solution w. An extensive treatment is presented in [58], [59]. A rather simple and explicit (negative) example is provided by Graham in [33] for the case where Ω is the ball in R^n, $\rho(x) \equiv 1$, $A(x) \equiv I$, $\Gamma_1 = \Gamma = \partial\Omega$, $\Gamma_0 = \emptyset$. There it is shown that controls $u \in L^2(\Gamma \oplus [0, T])$ are capable of producing states $(w(\cdot, T), v(\cdot, T)$ $(\equiv \frac{\partial w}{\partial t}(\cdot, T)))$ which are not "finite energy" states, i.e. do not lie in $H^1(\Omega) \oplus H^0(\Omega)$. This "excessive" effect of controls u might seem to promise rather strong controllability results but, in fact, it leads to considerable difficulty in some theorems, as we shall point out in Section 6.

The basic "approximate controllability" result appears in [88], [89]. The approach was suggested to the author by J. L. Lions in a private correspondence and takes the same form as a comparable result for the heat equation proved in [57]. (More on this in the next section which treats the heat equation). As is usually the case with theorems of this type, it is proved in the context of the duality structure discussed in Section 2.

Referring again to Figure 2.1 and the preceding material in Section 2, we let

$$X = Z = \{(w, v) \in H^1(\Omega) \oplus H^0(\Omega) \,|\, w(x) = 0, \quad x \in \Gamma_0\} \tag{5.9}$$

with the Hilbert space structure induced by $H^1(\Omega) \oplus H^0(\Omega)$. The map $S: X \to Z$ is defined for a fixed $T > 0$ by

$$S(w_0, v_0) = (\tilde{w}(\cdot, T), \tilde{v}(\cdot, T)) , \tag{5.10}$$

$\tilde{w}(\cdot, T)$, $\tilde{v}(\cdot, T)$ $(= \frac{\partial w}{\partial t}(\cdot, T))$ being the state achieved at time $t = T$ by the generalized solution $\tilde{w}$, and its time derivative determined by (5.1), (5.5), (5.6) (with $u \equiv 0$), (5.7) and (5.8) (with $m = 1$). We let

$$Y = L^2(\Gamma_1 \times [0, T])$$

and define

$$Cu = (\hat{w}(\cdot, T), \hat{v}(\cdot, T)) \tag{5.11}$$

where $\hat{w}$ is the solution of (5.1), (5.5), (5.6) with

$$\hat{w}(\cdot, 0) \equiv \hat{v}(\cdot, 0) \, (\equiv \frac{\partial \hat{w}}{\partial t}(\cdot, 0)) \equiv 0 . \tag{5.12}$$

We have remarked that we cannot be certain that $(\hat{w}(\cdot, T), \hat{v}(\cdot, T)) \in Z$ for arbitrary $u \in L^2(\Gamma_1 \times [0, T])$. Hence C is, in general, an unbounded operator, as anticipated in Section 2. We take the domain of C, $\mathcal{D}(C)$, to be the subspace of $Y = L^2(\Gamma_1 \times [0, T])$ consisting of u for which $\hat{w}(\cdot, T)$, $\hat{v}(\cdot, T)$ does, indeed, lie in Z, as defined by (5.9) (note that S and C are constructed just as in Section 3, but in that section C was a bounded operator). We remark that the fact that $\mathcal{D}(C)$ is dense in Y follows from the fact (see [88], [59]) that $\mathcal{D}(C)$ contains all functions $u \in C^\infty(\Gamma_1 \times [0, T])$ whose support is a closed subset of the interior of $\Gamma_1 \times [0, T]$.

Now let $y = y(x, t)$ be a solution in Z of

$$\rho(x) \frac{\partial^2 y}{\partial t^2} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(\alpha_j^i(x) \frac{\partial y}{\partial x^i}\right) = 0 \tag{5.13}$$

satisfying boundary conditions

$$y(x, t) \equiv 0, \quad x \in \Gamma_0 \tag{5.14}$$

$$\left(\nu(x),\ A(x) \frac{\partial y}{\partial x}(x, t)\right)_{E^n} = 0, \quad x \in \Gamma_1 \ . \tag{5.15}$$

We introduce the energy form on $X = Z$:

$$E(w(\cdot, t),\ v(\cdot, t)) = \tfrac{1}{2} \int_\Omega \left[\rho(x)\, v(x, t)^2 + \left(\frac{\partial w}{\partial x}(x, t),\ A(x) \frac{\partial w}{\partial x}(x, t)\right)_{E^n}\right] dx \tag{5.16}$$

and the related "energy inner product"

$$\langle (w(\cdot, t),\ v(\cdot, t));\ (y(\cdot, t),\ z(\cdot, t)) \rangle_E$$

$$= \int_\Omega \left[\rho(x)\ v(x, t)\ z(x, t) + \left(\frac{\partial w}{\partial x}(x, t),\ A(x) \frac{\partial y}{\partial x}(x, t)\right)_{E^n}\right] dx$$

for which $\sqrt{2E}$ is the associated norm. Letting w, $v = \frac{\partial w}{\partial t}$, y, $z = \frac{\partial y}{\partial t}$ satisfy (5.1), (5.5), (5.6) and (5.13), (5.14), (5.15), respectively, a formal computation (which can be justified, using, e.g., the methods in [59]) shows that

$$\langle (w(\cdot, T),\ v(\cdot, T));\ (y(\cdot, T),\ z(\cdot, T)) \rangle_E - \langle (w(\cdot, 0),\ v(\cdot, 0));\ (y(\cdot, 0), z(\cdot, 0)) \rangle_E$$

$$= \int_0^T \int_{\Gamma_1} z(x, t)\ u(x, t) ds\ dt \tag{5.17}$$

where ds is the element of surface area on Γ.

If we now replace (w, v) by $(\hat{w}, \hat{v})$, the solution of (5.1), (5.5), (5.6) with $\hat{w}(\cdot, 0) \equiv \hat{v}(\cdot, 0) \equiv 0$, we find that (5.17) becomes

$$\langle Cu, (y(\cdot, T), z(\cdot, T))\rangle_E = (u, z)_{L^2(\Gamma_1 \times [0, T])(= Y)} ,$$

which gives (C^* now being defined in terms of $\langle \, , \rangle_E$)

$$C^*(y(\cdot, T), z(\cdot, T)) = Z\Big|_{\Gamma_1 \times [0, T]} , \tag{5.18}$$

i.e., C^* associates with the terminal state $(y(\cdot, T), z(\cdot, T))$ the observation which is the restriction to $\Gamma_1 \times [0, T]$ of $z(x, t) = \frac{\partial y}{\partial t}(x, t)$, y being the solution of (5.13), (5.14), (5.15) with the indicated terminal state $(y(\cdot, T), z(\cdot, T))$. *The domain of* C^* *consists of those* $(y(\cdot, T)$, $z(\cdot T))$ for which $z \in L^2(\Gamma_1 \times [0, T])$. (Note that the "trace theorem" ([1], [58]) only gives $z \in \mathcal{C}([0, T]; H^{-\frac{1}{2}}(\Gamma_1))$.)

If we let (w, v) be $(\tilde{w}, \tilde{v})$, the solution of (5.1), (5.5), (5.6), (5.7), (5.8) with $m = 1$, $u \equiv 0$, we find that

$$\langle S(w_0, v_0); (y(\cdot, T), z(\cdot, T))\rangle_E = \langle (w_0, v_0); (y(\cdot, 0), z(\cdot, 0))\rangle_E$$

so that

$$S^*(y(\cdot, T), z(\cdot, T)) = (y(\cdot, 0), z(\cdot, 0)) . \tag{5.19}$$

We should remark here that if Γ_0 has a non-empty interior the energy norm $\sqrt{2E}$ (cf. (5.16) ff.) is equivalent to the induced norm in $Z \subseteq H^1(\Omega) \oplus H^0(\Omega)$ and Z, with this norm, is a Hilbert space $\mathcal{H}_E$.

When Γ_0 is empty, the important case wherein (5.6) applies on all of $\Gamma = \partial\Omega$, then the energy (5.16) vanishes for $w =$ constant, $v = 0$ and $\sqrt{2E}$ is only a semi-norm. If, however, we identify states which differ by a state $(c, 0)$, c constant, then $\sqrt{2E}$ is a norm and the resulting Hilbert space, which we shall still call $\mathcal{H}_E$, is equivalent to the orthogonal complement of $(c, 0)$ in $Z \subseteq H^1(\Omega) \oplus H^0(\Omega)$. These conventions will be tacitly assumed in the sequel. In any event the adjoints S^*, C^* are well defined, modulo these conventions, and if they are confused with the adjoints defined relative to the usual inner product in $H^1(\Omega) \oplus H^0(\Omega)$ little damage will result.

Approximate controllability of (5.1), (5.5), (5.6), i.e. of $\{X, Y, Z, S, C\}$, is assured if we establish observability of $\{C^*, S^*, Z, Y, X\}$. In the current context this means we should show:

$$\left.\begin{aligned} & z\Big|_{\Gamma_1 \times [0,T]} = 0 \text{ in } L^2(\Gamma_1 \times [0, T]) \\ & \Rightarrow S^*(y(\cdot, T), z(\cdot, T)) = (y(\cdot, 0), z(\cdot, 0)) = 0 \text{ in } \mathcal{H}_E \,. \end{aligned}\right\} \qquad (5.20)$$

If this is true then (5.1), (5.5), (5.6) is approximately controllable in the sense that each initial state (w_0, v_0) can be steered arbitrarily close, in the energy norm, to the zero state at time T, using a control in $L^2(\Gamma_1 \times [0, T])$. However, since this system is time reversible, one readily sees that this is equivalent to the question as to whether one can steer from any state $(w_0, v_0) \in Z$ to a state arbitrarily close to any other

$(w_1, v_1) \in Z$ with respect to the topology of $\mathscr{K}_E (= Z$ with the E norm, as noted earlier).

Let us stop and consider what (5.20) means. We are given a solution $y = y(x,t)$ of (5.13), (5.14), (5.15). Condition (5.15) is

$$(\nu(x),\ A(x)\frac{\partial y}{\partial x}(x,t))_{E^n} = 0, \quad x \in \Gamma_1, \quad 0 \leq t \leq T$$

while the premise of (5.20) is

$$z(x,t) = \frac{\partial y}{\partial t}(x,t) = 0, \quad x \in \Gamma_1, \quad 0 \leq t \leq T \ . \tag{5.21}$$

If we differentiate (5.15), formally, with respect to t, we have, in its place,

$$(\nu(x),\ A(x)\frac{\partial z}{\partial x}(x,t))_{E^n} = 0, \quad x \in \Gamma_1, \quad 0 \leq t \leq T \ . \tag{5.22}$$

Now (5.21), (5.22) constitute zero Cauchy data for the solution $z \equiv \frac{\partial y}{\partial t}$ of (5.13) on $\Gamma_1 \times [0,]$. The question is then seen to be one of uniqueness; if these Cauchy data are zero is it true that the solution z is zero?

The fact that the data (5.21), (5.22) are given on the "time-like" (cf. [11]) surface $\Gamma_1 \times [0,T]$ does not permit us to use the usual existence and regularity results for hyperbolic initial-boundary value problems. However, our problem is not one that has been historically neglected as it falls into a class of uniqueness problems studied by Holmgren [40], John [43] and others. A strengthened and modernized version of these uniqueness theories appears in Hormander's book [41].

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To describe these results adequately it will be necessary to introduce the concept of a characteristic surface for (5.13). Let $\phi(x,t)$ be a continuously differentiable function of (x,t) and let Σ be the surface

$$\Sigma = \phi(x,t) = 0 \quad .$$

For points $(x,t) \in \Sigma$ consider the "characteristic form"

$$\chi = \rho(x) \left(\frac{\partial\phi}{\partial t}\right)^2 - \left(\frac{\partial\phi}{\partial x}, A(x)\frac{\partial\phi}{\partial x}\right)_{E^n} ,$$

$\frac{\partial\phi}{\partial x}$ denoting the gradient of ϕ with respect to x. We say that Σ is

characteristic if $\chi \equiv 0$ on Σ ;

space-like if $\chi > 0$ on Σ ;

time-like if $\chi < 0$ on Σ .

Consider now the set $K(\Gamma_1, T)$ defined as follows: $(x,t) \in K(\Gamma_1, T)$ if (x,t) lies in the interior of a "lens-shaped" region whose boundary consists entirely of a portion of $\Gamma_1 \times [0,T]$ and a portion of a second (arbitrary) time like surface Σ . It can be seen that

(a) $\partial K(\Gamma_1, T) - (\Gamma_1 \times [0,T])$ consists of characteristic surfaces;

(b) $K(\Gamma_1, T)$ is symmetric about the plane $t = T/2$;

(c) $K(\Gamma_1, T) \cap (\Omega \oplus \{T/2\})$ includes all points $(x, T/2)$, $x \in \Omega$, for which the distance from x to Γ_1, relative to paths in Ω, is less than $\min_{x\in\Omega} \left(\frac{\lambda_1(x)}{\rho(x)}\right)^{\frac{1}{2}} \frac{T}{2}$, where $\lambda_1(x)$ is the smallest eigenvalue of $A(x)$, $x \in \Omega$;

(d) $K(\Gamma_1, T)$ does not include any points $(x, T/2)$ of $\Omega \times \{T/2\}$

for which the "Ω-distance from x to Γ_1 exceeds $\max_{x \in \Omega} \left(\frac{\lambda_n(x)}{\rho(x)}\right)^{\frac{1}{2}} T/2$, where $\lambda_n(x)$ is the largest eigenvalue of $A(x)$;

(e) $K(\Gamma_1, T)$ varies continuously with respect to T (with respect to the Haussdorff metric, e.g.)

Some descriptive sketches are given in [88].

From the above we can infer that there is a least time T_1 for which

$$K(\Gamma_1, T) \cap (\Omega \oplus \{T/2\}) = \Omega \oplus \{T/2\}, \quad T > T_1 \ .$$

Now the above cited Holmgren-John uniqueness theory includes the following result

Theorem 5.1. *If* x *is a generalized solution of* (5.13) *satisfying* (5.21) *and* (5.22) *then*

$$z \equiv 0, \quad (x,t) \in K(\Gamma_1, T) \ .$$

Consider now the observation (equivalently, the control) problem for a time $T > T_1$, T_1 as defined above. Choosing ε so that $T_1 + 2\varepsilon = T$ we can see that

$$K(\Gamma_1, T) \supseteq \oplus (T/2 - \varepsilon, T/2 + \varepsilon)$$

and from this it is possible to conclude that

$$z(x, T/2) \equiv \frac{\partial z}{\partial t}(x, T/2) \equiv 0, \quad x \in \Omega \ .$$

The usual uniqueness results then show that

$$z(x,t) \equiv 0, \quad x \in \Omega, \quad t \in [0,T] \ .$$

Since we have taken $z \equiv \frac{\partial y}{\partial t}$, we have

$$\frac{\partial y}{\partial t}(x, t) \equiv 0, \quad x \in \Omega, \quad t \in [0, T] .$$

Then, clearly

$$y(x, t) \equiv y(x), \quad x \in \Omega, \quad t \in [0, T] .$$

Since y is a solution, at least in the generalized sense, of (5.13) we now have

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\left(\alpha_j^i(x) \frac{\partial y}{\partial x^j}\right) = 0, \quad x \in \Omega \tag{5.23}$$

together with boundary conditions of the form (5.14), (5.15) for y. If Γ_0 (where $y(x) \equiv 0$) has a non-empty interior, the elliptic boundary value problem (5.23), (5.14), (5.15) has only the solution $y(x) \equiv 0$. If Γ_0 is empty, however, $y(x) \equiv c$, with c any constant, is also a solution.

In any event we have seen that for $T > T_1$ the condition

$$\begin{aligned} C^*(y(\cdot, T), z(\cdot, T)) &= Z\Big|_{\Gamma_1 \otimes [0,T]} \left(= \frac{\partial y}{\partial t}\Big|_{\Gamma_1 \otimes [0,T]}\right) \\ &= 0 \end{aligned} \tag{5.24}$$

implies

$$y(x, 0) = c, \ \frac{\partial y}{\partial t}(x, 0) = 0, \quad x \in \Omega ,$$

where c is some constant, equal to zero if Γ_0 has a non-empty interior. Thus, in every case we have

$$\|(y(\cdot, 0), \quad z(\cdot, 0))\|_{\mathcal{H}_E} = 0 ,$$

i.e., the implication (5.20) is valid. Thus $\{C^*, S^*, Z, Y, X\}$ is distinguishable and $\{X, Y, Z, S, C\}$ is approximately controllable. When Γ_0 has a non-empty interior the latter means that (5.1), (5.5), (5.6) is approximately controllable in the space $X = Z$ with the $H^1(\Omega) \oplus H^0(\Omega)$ topology. When Γ_0 is empty this approximate controllability means control modulo constant states, $\frac{\partial w}{\partial t} \equiv 0$, $w \equiv c$. Almost certainly a more refined analysis would show, in fact, that if $T > T_1$ the constant component of the solution can be controlled as well but this has not been done, to the authors knowledge, except for the case where $\rho(x) \equiv 1$, $A(x) \equiv I$ and Ω is a ball in E^n ([33]).

To save space we mention only briefly the cases $T = T_1$ and $T < T_1$. For $T < T_1$ we have neither distinguishability of (5.13), (5.14), (5.15) via the observation (5.18) nor approximate controllability of (5.1), (5.5), (5.6). The proof appears in [88], [89]. For $T = T_1$ the situation is rather complex as the mutual geometry of Γ_1 and Ω plays a decisive role. For example, if Ω is the unit ball in R^n, $\rho(x) \equiv 1$, $A(x) \equiv I$ and $\Gamma_1 = \partial\Omega$ (Γ_0 empty) we do have approximate controllability for $T = T_1$. But if Ω is the rectangle $|x^i| \leq 1$, $i = 1, 2, \ldots, n$ and Γ_1 consists of two faces, e.g. the faces $x^1 = \pm 1$, we do not have approximate controllability for $T = T_1$. The mathematical discussion appears in [89].

To summarize, we have

Theorem 5.2. There is a $T_1 > 0$ such that the system (5.1), (5.5), (5.6) is approximately controllable in $\mathscr{X}_E$ (Z with the E topology) using controls $u \in L^2(\Gamma_1 \times [0, T])$, provided $T > T_1$. If $T < T_1$ the system is not approximately controllable. If $T = T_1$ approximate controllability may or may not obtain, depending on the mutual geometry of Ω and Γ_1 (details in [89]).

The parallel distinguishability result for (5.13), (5.14), (5.15) with the observation (5.18) is the validity of (5.20) for $T > T_1$, etc.

The above approximate controllability result has been applied to resolve certain stabilizability problems. We will discuss this further in Section 6. Nevertheless it is a rather weak result with meagre consequences – as one should expect from the ease with which it is proved. Exact controllability results come much harder but are worth a lot more, mathematically, once one has obtained them. They are our next order of business.

We consider again the system (5.1), (5.5), (5.6) but we restrict attention to the case $\rho(x)$, $A(x)$ constant. It is then easy to see, via a change of variables ([93]), that we may as well treat only

$$\frac{\partial^2 w}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2 w}{(\partial x^i)^2} = 0 \tag{5.25}$$

the traditional " wave equation". The boundary conditions become

$$w(x, t) \equiv 0, \quad x \in \Gamma_0, \quad t \geq 0 \tag{5.26}$$

$$\frac{\partial w}{\partial \nu}(x, t) \equiv u(x, t), \quad x \in \Gamma_1, \quad t \geq 0 \tag{5.27}$$

where $\frac{\partial w}{\partial \nu}$ denotes the directional derivative along the normal direction ν to the boundary Γ of Ω.

We shall require the concept of a "star-shaped" region in what follows. A region $\Omega^* \subseteq R^n$ is star shaped if there is a point $x^* \in \Omega^*$ such that for every $x \in \Omega^*$ the line segment joining x and x^* also lies in Ω^*. A further notion which we shall use is that of a "star-complemented" pair. We consider the region $\Omega \subseteq R^n$ with boundary $\partial\Omega = \Gamma$ and we let Γ_1 denote a relatively open subset of Γ. We say that the pair (Ω, Γ_1) is star-complemented if there exists a star shaped region Ω^* with boundary $\partial\Omega^* = \Gamma^*$ such that

$$\Omega \subseteq \overline{\Omega^*}^{\,c} \qquad (\overline{}\ \text{closure}, \ ^c\ \text{complement})$$

$$\Gamma_1 = \Gamma - \Gamma^*, \quad \text{i.e.} \quad \Gamma - \Gamma_1 \subseteq \Gamma^* .$$

(See the geometric configuration displayed in Fig. 5.1.) An equivalent statement would be to say that (Ω, Γ_1) is star complemented if there is a point $x^* \notin \overline{\Omega}$ with the property that each point $x \in \Gamma_0 = \Gamma - \Gamma_1$ can be connected to x^* by a line segment which, except for the point x itself, lies entirely outside $\overline{\Omega}$.

It is also necessary to assume certain "regularity" conditions concerning the manner in which Ω meets Ω^*. These conditions permit one to employ the "reflection" methods described in [58] in order to carry out the extension procedure to be described in the proof of the next theorem. It would carry us too far astray to detail these assumptions here. The interested reader is referred to [93].

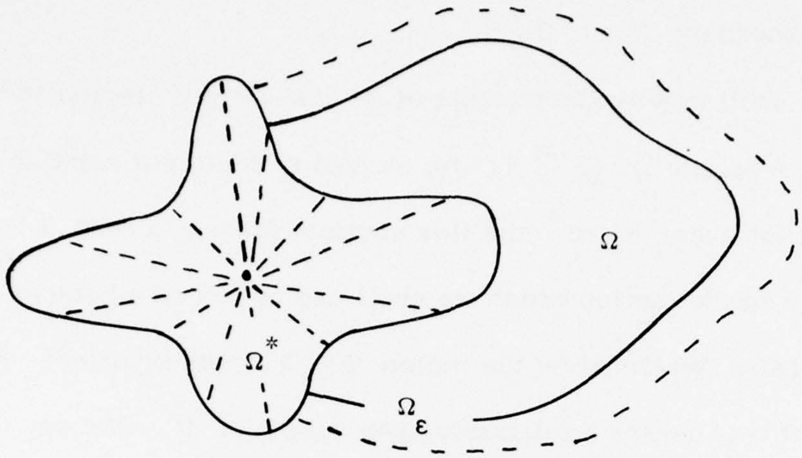

Figure 5.1. The domains Ω, Ω^*, Ω_ε

The main (exact) controllability result is

Theorem 5.3. Consider the system (5.25), (5.26), (5.27) in an open, bounded connected region $\Omega \subseteq R^n$ with boundary Γ having Γ_1 as a relatively open subset. If the pair Ω, Γ_1 is star-complemented then this system is exactly controllable in the space (with induced Hilbert space structure)

$$\mathcal{H}_0^2(\Omega) = \{(w, v(=\frac{\partial w}{\partial t})) \in H^2(\Omega) \oplus H^1(\Omega) \mid w(x) =$$

$$v(x) = 0, \quad x \in \Gamma - \Gamma_1 = \Gamma_0\} \tag{5.28}$$

in the sense that there is a positive number T_1 such that if $T > T_1$, and one is given arbitrary states (w_0, v_0), $(w_T, v_T) \in \mathcal{H}_0^2$, then there is a control $u \in L^2(\Gamma_1 \times [0, T)$ such that the solution of (5.25), (5.26), (5.27) and

$$w(\cdot, 0) = w_0,\ v(\cdot, 0) = \frac{\partial w}{\partial t}(\cdot, 0) = v_0 \tag{5.29}$$

is steered, by the action of the control u, to the terminal state

$$w(\cdot, T) = w_T,\ v(\cdot, T) = \frac{\partial w}{\partial t}(\cdot, T) = v_T\ . \tag{5.30}$$

The appearance of the space $\mathcal{H}_0^2(\Omega) \subseteq H^2(\Omega) \oplus H^1(\Omega)$ here is somewhat curious and immensely inconvenient. We will see in a later section that it is rather important to be able to control all finite energy states, i.e., all states $(w, v) \in \mathcal{H}_0^1(\Omega) \subseteq H^1(\Omega) \oplus H^0(\Omega)$ for which $w(x) = 0$, $x \in \Gamma_0 - \Gamma_1$. We will comment on this further after we sketch the proof.

Proof of Theorem 5.3 (Sketch). The method used here is a very simple one – completely "organic" in a sense, as the idea is to remove artificial barriers, let "nature take its course" and then emulate nature's methods. It is similar in form to the proof ([93]) of Theorem 2.10.

For some positive $\varepsilon > 0$ let

$$\Omega_\varepsilon = \{x \in R^n \mid x \notin \overline{\Omega^*},\ \ \|x - \hat{x}\| < \varepsilon$$
$$\text{for some } \hat{x} \in \Omega\},\ \text{(see Fig. 5.1)},$$

that is, Ω_ε is the "ε-neighborhood of Ω outside Ω^*". Let (cf. (5.28))

$$\mathcal{H}_\varepsilon^2(R^n - \Omega^*) = \{(w, v) \in H^2(R^n - \overline{\Omega^*}) \oplus H^1(R^n - \overline{\Omega^*}) \mid$$
$$w(x) = v(x) = 0,\ \ x \in \Gamma^* = \partial\Omega^*,\ \text{ and}$$
$$w(x) = v(x) = 0,\ \ x \notin \Omega_\varepsilon\}\ .$$

Now it can be shown (see [58], [93]) that there is a bounded linear transformation

$$F: \mathcal{H}_0^2(\Omega) \to \mathcal{H}_\varepsilon^2(R^n - \Omega^*)$$

with the following properties for $(\hat{w}, \hat{v}) \in \mathcal{H}_0^2(\Omega)$:

$$\begin{aligned} &F(\hat{w}, \hat{v}) \equiv (\hat{w}^e, \hat{v}^e) \in \mathcal{H}_\varepsilon^2(R^n - \Omega^*) ; \\ &\hat{w}^e(x) = \hat{w}(x), \quad \hat{v}^e(x) = \hat{v}(x), \quad x \in \Omega ; \\ &\|(\hat{w}^e, \hat{v}^e)\|_{\mathcal{H}_\varepsilon^2(R^n - \Omega^*)} \le K_F \|(\hat{w}, \hat{v})\|_{\mathcal{H}_0^2(\Omega)} \equiv \|F\| \, \|(\hat{w}, \hat{v})\|_{\mathcal{H}_0^2(\Omega)} . \end{aligned} \tag{5.31}$$

Thus F extends each $(\hat{w}, \hat{v}) \in \mathcal{H}_0^2(\Omega)$ to a pair of functions $(\hat{w}^e, \hat{v}^e)$ with comparable smoothness *in* $R^n - \overline{\Omega^*}$ *with the additional property that* $\hat{w}^e, \hat{v}^e$ vanish outside Ω_ε.

Now let $(\hat{w}_0, \hat{v}_0)$ be an arbitrary state in $\mathcal{H}_0^2(\Omega)$ and consider the equation (5.25) in the region $R^n - \overline{\Omega^*}$ *with initial state being the above* described extension:

$$(\hat{w}(\cdot, 0), \hat{v}(\cdot, 0)) = F(\hat{w}_0, \hat{v}_0) \equiv (\hat{w}_0^e, \hat{v}_0^e) . \tag{5.32}$$

Following the well-established theory for solutions of (5.25), let $\hat{w}$ be the solution of (5.32) for $x \in R^n - \overline{\Omega^*}$, $t \geq 0$, with these initial data and

$$\hat{w}(x, t) = 0, \quad x \in \overline{\Omega^*}, \quad t \geq 0 . \tag{5.33}$$

The behavior of the solution $\hat{w}$ in the region $R^n - \overline{\Omega^*}$ exterior to the star-shaped region Ω^* has been the object of a great deal of study by Lax, Phillips [53], [54], Morawetz [70], [71], [72], [53], Strauss [106], Ralston [80], [81] and others. It is still a field of considerable interest

with many questions unresolved to the present day. Nevertheless, from the existing results we know the following. If, as we have assumed, the support of the initial state (5.32) lies in a bounded region, such as our region Ω_ε, and if we let $\hat{w}^r$ denote the restriction of the solution $\hat{w}$ of (5.25), (5.32), (5.33) to another bounded region, such as our Ω, then there exist positive numbers M, γ such that for all $t \geq 0$

$$\|(\hat{w}^r(\cdot, t), \hat{v}^r(\cdot, t)\|_{\mathcal{K}^2_\varepsilon(R^n - \overline{\Omega^*}} \ . \tag{5.34}$$

(The results cited above actually establish (5.34) only with respect to the energy (equivalently, $H^1(\Omega) \oplus H^0(\Omega)$) topology but the extension to (5.34) for solutions initiating in $\mathcal{K}^2_0(\Omega)$ is immediate.) The basic idea is that all solutions whose supports initially lie in a bounded region, uniformly dissipate their energy "outward" into space away from the star-shaped region Ω^*. The theory has recently been extended, at least in R^2, to regions Ω^* which are not necessarily star-shaped but have the property of not "trapping rays" [72], [106]. An extension to hyperbolic systems has also been obtained [81].

Consider again the solution $\hat{w}$ of (5.25), (5.32), (5.33) for $x \in R^n - \overline{\Omega^*}$, $t \geq 0$, but with reference to its restriction to the surface $\Gamma_1 \times [0, \infty)$ which, by the star-complementation conditions, lies in $R^n - \overline{\Omega^*}$. More precisely, consider the behavior of $\frac{\partial \hat{w}}{\partial \nu}$ there. From the properties of w and the general trace theorem [1], [58] we know that, for any fixed $T > 0$,

$$\frac{\partial \hat{w}}{\partial \nu}\bigg|_{\Gamma_1 \times [0,T]} \in C([0,T]; H^{\frac{1}{2}}(\Gamma_1))$$

$$\subseteq L^2(\Gamma_1 \times [0,T])$$

and further, for some $B > 0$,

$$\left\|\frac{\partial \hat{w}}{\partial \nu}\right\|_{L^2(\Gamma_1 \times [0,T])} \leq B\|(\hat{w}_0^e, \hat{v}_0^e)\|_{\mathscr{V}_\varepsilon^2(R^n - \overline{\Omega^*})} \leq B\|F\|\,\|\hat{w}_0, \hat{v}_0\|_{\mathscr{V}_0^2(\Omega)} .$$

Having set all of this down we are ready to proceed. We choose T large enough so that (cf. (5.31), (5.34))

$$\|F\| M e^{-\gamma T} < 1 . \tag{5.35}$$

Then we take $(\hat{w}_0, \hat{v}_0) \in \mathscr{V}_0^2(\Omega)$ and

(a) extend to $(\hat{w}_0^e, \hat{v}_0^e) \in \mathscr{V}_\varepsilon^2(R^n - \overline{\Omega^*})$ via F;

(b) construct the solution $\hat{w}$ of (5.25), (5.32), (5.33) with $(\hat{w}(\cdot, 0), \hat{v}(\cdot, 0)) = \hat{w}_0^e, \hat{v}_0^e$ as the initial state;

(c) set $\hat{u}(x,t) = \frac{\partial \hat{w}}{\partial \nu}(x,t), \quad x \in \Gamma_1, \quad t \in [0,T]$.

We note then that $\hat{u}$, $\hat{w}^r$ and $\hat{v}^r \left(= \frac{\partial \hat{w}^r}{\partial t}\right)$ jointly satisfy (5.25), (5.29), (5.26), (5.27) and, by the usual uniqueness theorems, constitute the only solution of that mixed initial initial-boundary value problem. We note that (5.35) gives

$$\|\hat{w}^r(\cdot, T), \hat{v}^r(\cdot, T)\|_{\mathscr{V}_0^2(\Omega)} \leq M e^{-\gamma T}\|(\hat{w}_0^e, \hat{v}_0^e)\|_{\mathscr{V}_\varepsilon^2(R^n - \overline{\Omega^*})}$$

$$\leq \|F\| M e^{-\gamma T}\,\|(\hat{w}_0, \hat{v}_0)\|_{\mathscr{V}_0^2(\Omega)} . \tag{5.36}$$

Now let

$$(\tilde{w}_1, \tilde{v}_1) = -(\hat{w}^r(\cdot, T), \hat{v}^r(\cdot, T)) \quad , \tag{5.37}$$

$$(\tilde{w}_1^e, \tilde{v}_1^e) = F(\tilde{w}_1, \tilde{v}_1) \quad . \tag{5.38}$$

that is, we extend the negative of the terminal state obtained by the process (a), (b), (c) above to a state $(\tilde{w}_1^e, \tilde{v}_1^e)$ which, like $(\hat{w}_0^e, \hat{v}_0^e)$, lies in $\mathscr{K}_\varepsilon^2(R^n - \overline{\Omega^*})$. Then we run the whole "(a), (b), (c)" process in reverse from $t = T$ to $t = 0$. We let $\tilde{w}$ be the solution of (5.25), (5.33) with terminal state $(\tilde{w}, \tilde{v})$ at time T in $R^n - \overline{\Omega^*}$, $t \leq T$. We let $\tilde{u}(x, t) = \frac{\partial \tilde{w}}{\partial \nu}(x, t)$, $x \in \Gamma_1$, $0 \leq t \leq T$ and again have $\tilde{u} \in L^2(\Gamma_1 \times [0, T])$. Then $\tilde{u}, \tilde{w}^r, \tilde{v}^r$, the last twobeing the restriction of $\tilde{w}, \tilde{v}$ to $x \in \Omega$, jointly satisfy (5.25), (5.26), (5.27). We let

$$(\tilde{w}_0, \tilde{v}_0) = (\tilde{w}^r(\cdot, 0), \tilde{v}^r(\cdot, 0)) \quad .$$

Now, since (5.25), (5.33) are invariant under time reversal, the decay properties of $\tilde{w}$ backward in time are the same as those of $\hat{w}$ forward in time. Thus we now have using (5.35), (5.36),

$$\|(\tilde{w}_0, \tilde{v}_0)\|_{\mathscr{K}_0^2(\Omega)} \leq \|F\| M e^{-\gamma T} \|(\hat{w}^r(\cdot, T), \hat{v}^r(\cdot, T))\|_{\mathscr{K}_0^2(\Omega)}$$

$$\leq \|F\|^2 M^2 e^{-2\gamma T} \|(\hat{w}_0, \hat{v}_0)\|_{\mathscr{K}_0^2(\Omega)} \quad . \tag{5.39}$$

Let

$$w(x, t) = \hat{w}^r(x, t) + \tilde{w}^r(x, t), \; x \in \Omega, \quad 0 \leq t \leq T \quad .$$

From the material above we see that w satisfies (5.25), (5.26) and

$$(w(\cdot\,,0),\ v(\cdot\,,0)) = (\hat{w}_0, \hat{v}_0) + (\tilde{w}_0, \tilde{v}_0) \tag{5.40}$$

$$\frac{\partial w}{\partial \nu}(x,t) = \frac{\partial \hat{w}}{\partial \nu}(x,t) + \frac{\partial \tilde{w}}{\partial \nu}(x,t) = \hat{u}(x,t) + \tilde{u}(x,t) \equiv u(x,t)$$

$$x \in \Gamma_1, \quad 0 \le t \le T\ ,$$

and, because of (5.37),

$$w(x,T) \equiv v(x,T) \equiv 0, \quad x \in \Omega\ .$$

In other words, the control $u = \hat{u} + \tilde{u}$ steers the solution w of (5.25), (5.26), (5.27) from the initial state (5.40) to zero. Thus, if (5.40) can be an arbitrary state in $\mathscr{H}_0^2(\Omega)$ we have exact null controllability in $\mathscr{H}_0^2(\Omega)$, that is, we can steer any point in $\mathscr{H}_0^2(\Omega)$ to zero in time T. Now this, in fact, is the situation. For $(\tilde{w}_0, \tilde{v}_0)$ is obtained from $(\hat{w}_0, \hat{v}_0)$ by a *linear map*, call it K. The requirement that (5.29) and (5.40) should agree is then

$$(I + K)\,(\hat{w}_0, \hat{v}_0) = (w_0, v_0)\ . \tag{5.41}$$

Since (5.39) shows that $\|K\| < 1$, (5.41) is uniquely solvable for $(\hat{w}_0, \hat{v}_0) \in \mathscr{H}_0^2(\Omega)$ and (5.29) may thus be satisfied by appropriate choice of $(\hat{w}_0, \hat{v}_0)$.

To complete our proof we merely note that the invariance of (5.25), (5.26), (5.27) under time reversal shows the problem of controlling from a zero initial state to an arbitrary terminal state $(w_1, \mathbf{v}_1)$ to be equivalent to the null controllability problem solved above. Then simply adding the two solutions we solve the problem as originally posed and the proof is complete.

We can now complete the remarks made following the statement of our theorem. Our restriction to states in $H^2(\Omega) \oplus H^1(\Omega)$ is made necessary by the very nature of the trace theorem, of which we clearly make essential use. For such states the solutions $\hat{w}$, $\tilde{w}$ lie in $\mathcal{C}([0,T]; H_0^2(R^n - \bar{\Omega}^*))$ and the truce theorem gives $\frac{\partial \hat{w}}{\partial \nu}$, $\frac{\partial \tilde{w}}{\partial \nu}$, and consequently the control u, as elements of $\mathcal{C}([0,T]; H^{\frac{1}{2}}(\Gamma_1 \times [0,T])$. If we use the "finite energy" space $H^1(\Omega) \oplus H^0(\Omega)$ we get $u \in \mathcal{C}([0,T]; H^{-\frac{1}{2}}(\Gamma_1))$ by the same process and u is a distribution of a particular type rather than a square integrable function. This might appear to be merely a matter of mathematical nit-picking but, *in fact, it is not. We will see* why in Section 6.

We actually conjecture that Theorem 5.2 is true with $H^1(\Omega) \oplus H^0(\Omega)$ used instead of $H^2(\Omega) \oplus H^1(\Omega)$, provided, of course, that the star-complementation condition remains in force. We also conjecture that in general it is not true with either of these spaces when the star complementation condition fails. Proof, or disproof, of these conjectures is, arguably, one of the most significant currently unsolved problems in the theory of control of partial differential equations. However, despite the absence of complete proofs, we are not without persuasive support for the validity of our conjectures. This also will be made clear in Section 6.

We conclude this section by noting (at least) a recent development *which seems to shed some light on the method of proof* which we have used in Theorem 5.3. Recent work due to Baras, Brockett, Helton ([3], [4], [35], [36], which we do not review here because it is primarily

related to the algebraic theories of representation of control systems, shows that there is a deep underlying relationship between the scattering theory of Lax, Phillips ([54]) and others and control theory. This makes the use of scattering theory results in the proof of Theorem 5.3 seem much less "accidental" than otherwise would be the case.

6. ENERGY DECAY FOR THE WAVE EQUATION WITH BOUNDARY DAMPING

Our intention in this section is to develop results comparable to Theorem 3.9, but now in the context of the wave equation

$$\rho(x) \frac{\partial^2 w}{\partial t^2} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}(\alpha_j^i(x) \frac{\partial w}{\partial x^j}) = 0 \tag{6.1}$$

for $t \geq 0$ and $x \in \Omega$, a domain in R^n as described in the previous section. Again we suppose $\Gamma = \partial\Omega$ divided into two regions: $\Gamma = \Gamma_0 \cup \Gamma_1$, with Γ_1 relatively open and non-empty. We suppose

$$w(x, t) = 0, \quad x \in \Gamma_0, \quad t \geq 0 \tag{6.2}$$

and, for $\beta > 0$, we impose the condition $(v(x, t) \equiv \frac{\partial w}{\partial t}(x, t))$

$$v(x, t) + \beta(\frac{\partial w}{\partial x}(x, t), A(x)\, \nu(x))_{E^n} = 0, \quad x \in \Gamma_1, \quad t \geq 0 \tag{6.3}$$

where, as before, ν denotes the exterior normal direction. When Ω is an interval in R^1, $\rho = A = 1$, and Γ_1 is one endpoint, say $x = 1$, (6.3) becomes

$$\frac{\partial w}{\partial t}(1, t) + \beta \frac{\partial w}{\partial x}(1, t) = 0$$

which after transformation (see (4.5)ff.) takes a form included under (3.64) of Assumption 3.8. In particular, this is a <u>dissipative</u> boundary condition; supposing an initial state (cf. (5.28))

$$(w(\cdot, 0), \; v(\cdot, 0) \,(= \frac{\partial w}{\partial t}(\cdot, 0))) = (w_0, v_0) \in \mathcal{K}_0^2(\Omega) \tag{6.4}$$

the resulting solution $w = w(x, t)$ of (6.1), (6.2), (6.3) lies in $\mathcal{C}[0, \infty)$; $\mathcal{K}_0^2(\Omega))$ and we may differentiate the energy expression (cf. (5.16)):

$$\frac{d}{dt} E(w(\cdot, t), v(\cdot, t) = \frac{d}{dt}\left(\int_\Omega \left[\rho(x)(v(x,t))^2 + \left(\frac{\partial w}{\partial x}(x,t),\ A(x)\frac{\partial w}{\partial x}(x,t)\right)_{E^n} \right] dx\right.$$

$$= \int_\Omega \left[\rho(x)\, v(x,t)\frac{\partial^2 w}{\partial t^2}(x,t) + \left(\frac{\partial^2 w(x,t)}{\partial t \partial x},\ A(x)\frac{\partial w}{\partial x}(x,t)\right)_{E^n} \right] dx$$

= (after use of the divergence theorem and (6.1))

$$= \int_\Gamma \left[v(x,t)\left(\frac{\partial w}{\partial x}(x,t),\ A(x)\,\nu(x)\right)_{E^n}\right] ds = \text{(using (6.2), (6.3))}$$

$$= -\frac{1}{\beta}\int_{\Gamma_1} (v(x,t))^2\, ds \le 0 \quad . \tag{6.5}$$

We have, then, a situation much like that of Theorem 3.9. The energy is non-increasing but it is entirely possible that $v(x, t) \equiv 0$, $x \in \Gamma_1$, for t in some interval of finite length, foreboding some difficulty in an attempt to prove that

$$\lim_{t \to \infty} E(w(\cdot, t),\ v(\cdot, t)) = 0 \quad . \tag{6.6}$$

Actually, (6.6) can be proved under no more assumptions than we have already made. We make direct use of the extended invariance principle [34] as enunciated by Hale together with Theorem 5.1 of the previous section. As at the beginning of Section 5, we assume Γ_1 non-empty and relatively open in $\partial\Omega$.

Theorem 6.1. Under the above assumptions on the system (6.1), (6.2), (6.3) we have (6.6) for each generalized solution $w = w(x, t)$ corresponding to an initial state

$$(w(\cdot,0),\ v(\cdot,0)) = (w_0, v_0) \in \mathcal{K}^1_0(\Omega)\ (= \{(w,v)$$

$$\in H^1(\Omega) \oplus H^0(\Omega),\ w(x) = 0,\ x \in \Gamma_0 .\ \text{Called Z in (5.9).}) \tag{6.7}$$

Sketch of Proof. Though we want to establish (6.6) for initial states (6.4) in $\mathcal{K}^1_0(\Omega)$, we shall first consider a solution $\tilde{w} = \tilde{w}(x,t)$ corresponding to an initial state (cf. (5.28))

$$(\tilde{w}(\cdot,0),\ \tilde{v}(\cdot 0)) = (\tilde{w}_0, \tilde{v}_0) \in \mathcal{K}^2_0(\Omega) . \tag{6.8}$$

Then $(\tilde{w}(\cdot,t),\ \tilde{v}(\cdot,t)) \in C([0,\infty); \mathcal{K}^2_0(\Omega))$ and, carrying out (6.5), we have $E(\tilde{w}(\cdot,t),\ \tilde{v}(\cdot,t))$ non-increasing and hence

$$\lim_{t\to\infty} E(\tilde{w}(\cdot,t),\ \tilde{v}(\cdot,t)) = E_\infty \geq 0 . \tag{6.9}$$

This next step is to show that $(\tilde{w}(\cdot,t),\ \tilde{v}(\cdot,t))$ remains in a bounded subset of $\mathcal{K}^2_0(\Omega)$. This is done (see [79]) by establishing that

$$E(\frac{\partial\tilde{w}}{\partial t}(\cdot,t),\ \frac{\partial^2\tilde{w}}{\partial t^2}(\cdot,t)) = E(\tilde{v}(\cdot,t),\ \frac{\partial\tilde{v}}{\partial t}(\cdot,t))$$

$$= \tfrac{1}{2}\int_\Omega \{[\frac{1}{\rho(x)} \sum_{i,j=1}^n \frac{\partial}{\partial x^i}(\alpha^i_j(x)\frac{\partial w}{\partial x^j}(x,t)]^2 + (\frac{\partial v}{\partial x}(x,t), A(x)\frac{\partial v}{\partial x}(x,t))_{E^n}\}dx$$

is non-increasing. Now it is known ([1]) that bounded subsets of $\mathcal{K}^2_0(\Omega)$ are pre-compact with respect to the topology (cf. (6.7)) of $\mathcal{K}^1_0(\Omega)$, which, as we have noted earlier, is derived from a norm whose square is essentially equivalent to the energy, E. From this one can establish that

$$\lim_{k\to\infty} \|\tilde{w}(\cdot,t_k),\ \tilde{v}(\cdot,t_k) - (\hat{w}_0, \hat{v}_0)\|_{\mathcal{K}^1_0(\Omega)} = 0$$

for some sequence $\{t_k\}$ with $\lim_{k\to\infty} t_k = \infty$ and for some $(\hat{w}_0, \hat{v}_0) \in \mathscr{V}_0^2(\Omega)$ (since the weak limit with respect to the $\mathscr{V}_0^2(\Omega)$ topology must agree with the strong limit in the $\mathscr{V}_0^1(\Omega)$ topology). One then shows that the solution $\hat{w} = \hat{w}(x,t)$ of (6.1), (6.2), (6.3) with

$$(\hat{w}(\cdot, 0), \hat{v}(\cdot, 0)) = (\hat{w}_0, \hat{v}_0)$$

has

$$E(\hat{w}(\cdot, t), \hat{v}(\cdot, t)) \equiv E_\infty \quad (\text{cf. } (6.9)), \quad t \geq 0 \tag{6.10}$$

$$\hat{v}(x,t) \equiv 0, \quad x \in \Gamma_1, \quad t \geq 0 \ . \tag{6.11}$$

But then Theorem 5.1 gives $\hat{v}(x,t) \equiv 0$ whence $E_\infty = 0$ and (6.9) gives

$$\lim_{t\to\infty} E(\tilde{w}(\cdot, t), \tilde{v}(\cdot, t)) = 0 \ .$$

Approximating the initial state $w_0 v_0$ in $\mathscr{V}_0^1(\Omega)$ by states $(\tilde{w}_0, \tilde{v}_0)$ in $\mathscr{V}_0^2(\Omega)$ *one readily obtains (6.6) for initial states in* $\mathscr{V}_0^1(\Omega)$ and the proof is complete.

It is clear that the scheme applied above is just that laid down in [34]. An earlier treatment than [79], the paper [104] by Slemrod, used similar methods to establish that

$$\lim_{t\to\infty} (w(\cdot, t), v(\cdot, t) = 0$$

in the weak topology of $\mathscr{V}_0^1(\Omega)$, making use of the weak pre-compactness of bounded subsets of that space.

Now the question arises: what more can we prove when the stronger controllability result of Theorem 5.3 obtains; i.e., when we consider the equation (5.25) ($\rho(x) \equiv 1$, $A(x) \equiv I$) and add the assumption that the pair (Ω, Γ_1) is star complemented? In this case our system is

$$\frac{\partial^2 w}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2 w}{(\partial x^i)^2} = 0, \quad x \in \Omega, \quad t \geq 0 , \tag{6.12}$$

$$w(x,t) = 0, \quad x \in \Gamma_1, \quad t \geq 0 . \tag{6.13}$$

$$\frac{\partial w}{\partial t}(x,t) + \beta \frac{\partial w}{\partial \nu}(x,t) = 0, \quad x \in \Gamma_1, \quad t \geq 0 . \tag{6.14}$$

It should be noted that one could obtain the above as a "closed loop system" realized via use of a linear feedback law

$$u(x,t) = -\frac{1}{\beta} \frac{\partial w}{\partial t}(x,t), \quad x \in \Gamma_1, \ t \geq 0$$

substituted in the control system (5.25), (5.26), (5.27) or one could envision this system arising naturally as the result of some dissipative mechanism, such as friction, acting on the portion Γ_1 of the boundary.

For $n = 1$ Theorem 3.9 applies to show that the energy decays exponentially, corresponding to the fact that finite energy states can be controlled with controls $u \in L^2[0,T]$. In the present section we are primarily concerned with the "higher dimensional" cases $n \geq 2$, where, as we shall see, the situation is decidedly more complicated.

If one takes further care in Section 5 one can show, in the context of the exact controllability results presented there, that for the control problem (5.25), (5.26), (5.27), (5.29), (5.30) we have

$$\|u\|_{L^2(\Gamma_1 \times [0,T])} \leq K(\|(w_0, v_0)\|^2_{\mathcal{H}^2_0(\Omega)} + \|(w_1, v_1)\|^2_{\mathcal{H}^2_0(\Omega)}) \tag{6.15}$$

for some positive constant K. It is also possible to see, and this will prove important in the sequel, that there is a constant $\hat{K} > 0$ such that

$$\left\|\frac{\partial w}{\partial t}\right\|^2_{L^2(\Gamma_1 \times [0,T])} \leq \hat{K}\left(\|(w_0, v_0)\|^2_{\mathcal{H}^2_0(\Omega)} + \|(w_1, v_1)\|^2_{\mathcal{H}^2_0(\Omega)}\right) \qquad (6.16)$$

The fact that we can prove the controllability only for states in $\mathcal{H}^2_0(\Omega)$, rather than in $\mathcal{H}^1_0(\Omega)$, has a definite weakening effect on what we can prove in regard to the decay of energy of solutions of (6.12), (6.13), (6.14).

Theorem 6.2. Let the pair (Ω, Γ_1) be star-complemented. Then every solution $w = w(x,t)$ of (6.12), (6.13), (6.14) satisfying an initial condition

$$(w(\cdot, 0), v(\cdot, 0)) = (w_0, v_0) \in \mathcal{H}^2_0(\Omega) \qquad (6.17)$$

has the property that the energy

$$E(w(\cdot, t), v(\cdot, t)) = \tfrac{1}{2} \int_\Omega \left[(v(x,t))^2 + \sum_{k=1}^n \left(\frac{\partial w}{\partial x_k}(x,t)\right)^2\right] dx$$

satisfies an inequality

$$E(w(\cdot, t), v(\cdot, t)) \leq \frac{C(w_0, v_0)}{1+t}, \quad t \geq 0 \qquad (6.18)$$

where $C(w_0, v_0)$ depends, in general, on $\|(w_0, v_0)\|_{\mathcal{H}^2_0(\Omega)}$.

Sketch of Proof. We proceed with a view to treating the system (6.12), (6.13), (6.14) in much the same way as we did the corresponding hyperbolic system in Theorem 3.9.

Let us suppose that we have a solution $w(x,t)$ of (6.12), (6.13), (6.14) corresponding to an initial state (6.17). The solution w then lies in $C([0,\infty); \mathcal{H}^2_0(\Omega)) \cap C^1([0,\infty); \mathcal{H}^1_0(\Omega))$ (see [58], [59]) and one may differentiate, as in (6.5), to obtain

$$\frac{d}{dt} E(w(\cdot, t), v(\cdot, t)) = -\frac{1}{\beta} \int_{\Gamma_1} (v(x, t))^2 ds \quad . \tag{6.18}$$

Letting $T \geq T_1$, the positive number introduced earlier in Theorem 5.3 we have

$$E(w_0, v_0) - E(w(\cdot, T), v(\cdot, T)) = \frac{1}{\beta} \int_0^T \int_{\Gamma_1} (v(x, t))^2 ds\, dt \quad . \tag{6.19}$$

We now return to the control problem which we treated in Theorem 5.3, specialized to a zero initial condition and a particular terminal condition defined in terms of the solution w described above:

$$\frac{\partial^2 y}{\partial t^2} - \sum_{k=1}^{n} \frac{\partial^2 y}{(\partial x^k)^2} = 0, \quad x \in \Omega, \quad t \geq 0 \tag{6.20}$$

$$y(x, t) = 0, \quad x \in \Gamma_0, \quad t \geq 0, \quad \frac{\partial y}{\partial \nu}(x, t) = u(x, t), \quad x \in \Gamma_1, \quad t \geq 0 \tag{6.21}$$

$$y(x, 0) = \frac{\partial y}{\partial t}(x, 0) = 0, \quad y(x, T) = w(x, T), \quad \frac{\partial y}{\partial t}(x, T) = \frac{\partial w}{\partial t}(x, T) \; ,$$

$$x \in \Omega \quad . \tag{6.22}$$

We have seen that we can find a solution $y(x, t)$, $x \in \Omega$, $0 \leq t \leq T$, for this problem. A short computation similar to (6.5) then shows that

$$2\, E(w(\cdot, T), v(\cdot, T)) = \int_\Omega \left(\frac{\partial y}{\partial t}(x, T) \frac{\partial w}{\partial t}(x, T) + \sum_{k=1}^{n} \frac{\partial y}{\partial x^k}(x, T) \frac{\partial w}{\partial x^k}(x, T)\right) dx$$

$$- \int_\Omega \left(\frac{\partial y}{\partial t}(x, 0) \frac{\partial w}{\partial t}(x, 0) + \sum_{k=1}^{n} \frac{\partial y}{\partial x^k}(x, 0) \frac{\partial w}{\partial x^k}(x, 0)\right) dx$$

$$= \int_0^T \int_{\Gamma_1} \left(\frac{\partial w}{\partial t}(x, t)\, u(x, t) + \frac{\partial y}{\partial t}(x, t) \frac{\partial w}{\partial \nu}(x, t)\right) ds\, dt = (\text{cf. } (6.14))$$

$$= \int_0^T \int_{\Gamma_1} \left[\frac{\partial w}{\partial t}(x, t)\left(u(x, t) - \frac{1}{\beta} \frac{\partial y}{\partial t}(x, t)\right]\right. ds\, dt \; . \tag{6.23}$$

Applying the Schwartz inequality we find

$$\int_0^T \int_{\Gamma_1} \left(\frac{\partial w}{\partial t}(x,t)\right)^2 ds\, dt \geq \frac{2\, E(w(\cdot,T),\ v(\cdot,T))^2}{\int_0^T \int_{\Gamma_1} \left(u(x,t) - \frac{1}{\beta}\frac{\partial y}{\partial t}(x,t)\right)^2 ds\, dt}\ .$$

$$\geq \frac{E(w(\cdot,T),\ v(\cdot,T))^2}{\|u\|^2_{L^2(\Gamma_1\times[0,T])} + \left(\frac{1}{\beta}\right)^2 \left\|\frac{\partial y}{\partial t}\right\|^2_{L^2(\Gamma_1\times[0,T])}}$$

$\geq$ (cf. (6.15), (6.16), (6.22))

$$\geq \frac{E(w(\cdot,T),\ v(\cdot,T))^2}{M\|(w(\cdot,T),\ v(\cdot,T))\|^2_{\mathcal{H}_0^2(\Omega)}} \tag{6.24}$$

for some $M > 0$, independent of the particular solution w. Consequently we now have, from (6.19),

$$E(w_0, v_0) - E(w(\cdot,T),\ v(\cdot,T)) \geq \frac{E(w(\cdot,T),\ v(\cdot,T))^2}{M\|(w(\cdot,T),\ v(\cdot,T))\|^2_{\mathcal{H}_0^2(\Omega)}}\ . \tag{6.25}$$

The same sort of argument applied on successive intervals $[kT, (k+1)T]$ shows that

$$E(w(\cdot,kT),\ v(\cdot,kT)) - E(w(\cdot,(k+1)T),\ v(\cdot,(k+1)T))$$

$$\geq \frac{E(w(\cdot,(k+1)T),\ v(\cdot,(k+1)T))^2}{M\|(w(\cdot,(k+1)T),\ v(\cdot,(k+1)T))\|^2_{\mathcal{H}_0^2(\Omega)}}\ . \tag{6.26}$$

Now, as noted in Theorem 6.1, a slightly more careful study of (6.12), (6.13), (6.14) (see [79] for details) enables one to see that

$$\|(w(\cdot,t),\ v(\cdot,t))\|^2_{\mathscr{H}_0^2(\Omega)} \leq B\|(w_0,\ v_0)\|^2_{\mathscr{H}_0^2(\Omega)},\quad t \geq 0\ . \tag{6.27}$$

Letting $\varepsilon_k = E(w(\cdot,kT),\ v(\cdot,kT))$, $k = 0,1,2,\ldots$, (6.25), (6.26) and (6.27) combine to show that

$$\varepsilon_k - \varepsilon_{k+1} \geq \tilde{c}(w_0,\ v_0)\ \varepsilon^2_{k+1},\quad k = 0,1,2,\ldots \tag{6.28}$$

where $\tilde{c}$ is a positive constant depending on the initial state $(w_0,\ v_0)$ $\in \mathscr{H}_0^2(\Omega)$. Thus the ε_k satisfy a quadratic nonlinear recursion equation. From (6.28) one can establish ([79]) that, for some $c(w_0, v_0) \geq 0$,

$$\varepsilon_k \leq \frac{C(w_0,\ v_0)}{k+1},\quad k = 0,1,2,\ldots$$

and, making use of the fact that $E(w(\cdot,t),\ v(\cdot,t))$ is non-increasing, we have, finally

$$E(w(\cdot,t),\ v(\cdot,t)) \leq \frac{C(w_0,\ v_0)}{1+t},\quad t \geq 0\ ,$$

where $C(w_0,\ v_0)$ can be bounded in terms of $\|(w_0,\mathbf{v}_0)\|_{\mathscr{H}_0^2(\Omega)}$. This is just (6.18) and the theorem is proved.

We remark that if $C(w_0,\ v_0)$ could be bounded in terms of $\|(w_0,\ v_0)\|_{\mathscr{H}_0^1(\Omega)}$ (which is equivalent to the norm defined by $E(w_0,\ v_0)$), (6.18) would necessarily imply uniform exponential decay of energy as a result of the semigroup property of the solutions of (6.12), (6.13), (6.14). But this has not been shown to be the case and we must, for the moment, be content with the result we have.

As we have already noted in Section 5, we conjecture that, provided the pair (Ω, Γ_1) is star-complemented, Theorem 5.3 can be strengthened to give controllability in $\mathcal{H}_0^1(\Omega)$ and the result (6.18) can be strengthened to the extent of demonstrating uniform exponential decay of energy. Curiously enough, however, the improvement of (6.18) seems likely to be established first and then the strengthened version of Theorem 5.3, following a method similar to the proof in Section 5 but using (6.12), (6.13), (6.14) instead of the wave equation in the "external" region $\mathbb{R}^n - \overline{\Omega^*}$.

In [33] K. D. Graham and the present author have studied the control system (6.20), (6.21) for the special case where Ω is the unit ball in E^n, $\Gamma_1 = \Gamma = \partial\Omega$, $\Gamma_0 = \emptyset$. The method relies on separation of variables and reduction of the control problem to an infinite collection of moment problems similar to (4.15), but involving, in place of the σ_k, the various zeros $w_{k,j}$ of the Bessel functions of order j. We have been able to establish in this special case that all states $(w, v) \in \mathcal{H}^1(\Omega)$ $(= H^1(\Omega) \oplus H^0(\Omega))$ can be controlled with controls $u \in L^2(\partial\Omega \times [0, 2+\varepsilon])$, $\varepsilon > 0$. Curiously enough, however, the result does not lead to a proof of exponential decay for solutions of (6.12), (6.13), (6.14) in the corresponding situation. The problem is that it does not seem to be possible, via this method, to establish the inequality (6.16) with $\mathcal{H}^2(\Omega)$ replaced by $\mathcal{H}^1(\Omega)$ (though we do have (6.15) with $\mathcal{H}_0^2(\Omega)$ replaced by $\mathcal{H}^1(\Omega)$). This difficulty prevents adequate treatment of the denominator of the expression following the first inequality in (6.24).

There is one situation in which we can report results which back up our conjecture. That is the situation illustrated on the left of Figure 6.1 where Ω is a rectangle and Γ_1 consists of two (when $n = 2$) adjacent sides. This work is reported by J. P. Quinn and the present author in [79]. We take the equation

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \tag{6.29}$$

in the rectangle $0 < x < a$, $0 < y < b$, with boundary conditions

$$w(x, y, t) = 0, \quad x = 0 \text{ or } y = 0, \quad t \geq 0 \tag{6.30}$$

$$\frac{\partial w}{\partial t}(x, y, t) + \beta \frac{\partial w}{\partial \nu}(x, y, t) = 0, \quad x = a \text{ or } y = b, \quad t \geq 0$$

$$(\nu = x \text{ if } x = a, \quad \nu = y \text{ if } y = b) . \tag{6.31}$$

In this case one sees very easily that (Ω, Γ_1) (Ω = the rectangle, $\Gamma = \{(x, y) \mid x = a, \ 0 \leq y \leq b \text{ or } y = b, \ 0 \leq x \leq a\}$) is star-complemented.

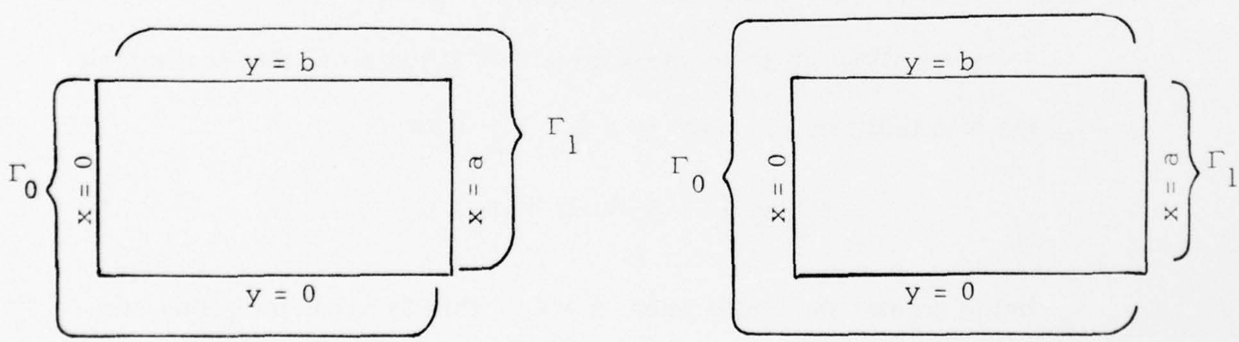

Figure 6.1. Two different configurations for the rectangle $0 \leq x \leq a$, $0 \leq y \leq b$.

Having set this up we take $\beta > 0$ but small and consider (6.29), (6.30), (6.31) to be a perturbation of (6.29), (6.30) and

$$\frac{\partial w}{\partial t}(x, y, t) = 0, \quad x = a \text{ or } y = b, \quad t \geq 0 \ . \tag{6.32}$$

Now for (6.29), (6.30), (6.32) the eigenfunctions of the infinitesimal generator of the relevant semigroup on $\mathcal{H}_0^1(\Omega) = \mathcal{H}_0^1([0, a] \times [0, b])$ can be computed explicitly. There are infinitely many eigenvalues σ, all of finite multiplicity, constituting a set Σ, which lie on the imaginary axis. There is also the eigenvalue 0, and it is of infinite multiplicity, the associated eigenspace consisting of states (w, v) with $v = 0$ and w a solution of

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0$$

$$w(x, y) = 0, \quad x = 0 \text{ or } y = 0 \qquad (\Gamma_0)$$

$$w(x, y) = g(x, y), \quad x = a \text{ or } y = b \qquad (\Gamma_2)$$

with $g \in H^{\frac{1}{2}}(\Gamma_1)$. This is the situation for $\beta = 0$.

As we allow β to increase from zero it turns out that the eigenvalues originally in Σ move to a set Σ_β with

$$\Sigma_\beta \subseteq \{z \in \mathbb{C} \mid \mathrm{Re}(z) < -\delta_\beta\} ,$$

δ_β being greater than zero when $\beta > 0$. This is not astonishing and it is precisely what we want. More surprising, perhaps, is the fact that the eigenvalue 0 of infinite multiplicity passes into a countably

infinite set Z_β of real, negative eigenvalues which, for any $\beta > 0$, no matter how small, have $-\infty$ as their only limit point. As a consequence it is seen that (6.29), (6.30), (6.31) cannot be time reversible when $\beta > 0$ – the process is dissipative in an essential way, just as heat conduction processes are.

However, the fact that all system eigenvalues for (6.29), (6.30), (6.31), $\beta > 0$ but small, are uniformly displaced into the left half plane allows one, after certain analysis of the eigenfunctions also, to establish that for this system one has

$$E(w(\cdot,t),\ v(\cdot,t)) \le Ce^{-\delta t}E(w(\cdot,0), v(\cdot,0)), \quad t \ge 0$$

with $C, \delta > 0$ and independent of the particular state $(w(\cdot,0),\ v(\cdot,0))$. That is, we have uniform exponential decay of energy in this case.

The results concerning decay of solutions of the wave equation in bounded subsets of $\overline{\Omega^*}^{c}$ referred to earlier in this section are usually obtained now (this was not true originally; see e.g., [70]) with the use of the method of geometrical optics. We have some hope that it may be possible to employ this method to study the decay of solutions of (6.12), (6.13), (6.14) when the star-complementation condition is satisfied. Careful attention will have to be paid to the reflection of waves and rays from the energy absorbing surface Γ_1. One aspect of the problem indicates caution. The method of geometrical optics is basically associated with wave-type phenomena. But the presence of real eigenvalues tending to $-\infty$ (the set Z_β) in the case of (6.29), (6.30), (6.31)

indicates that some processes which are not really of wave type may well be operative in the context of these systems.

Some indication that the star-complementation condition is necessary may be obtained by considering the situation displayed on the right of Fig. 6.1, where we again consider (6.29) but now suppose that (6.30) holds on the three sides there indicated as Γ_0 while (6.31) holds only on the single side $x = a$. In this case it is shown in [79] that as β is increased from 0 the set of eigenvalues Σ_β is not uniformly displaced from the imaginary axis – a subsequence of eigenvalues asymptotically approaches it. In this case there can be no uniform exponential decay and there can be no controllability in $\mathcal{H}^1_0(\Omega)$ (since the latter would imply the former). Whether the weaker result of controllability in $\mathcal{H}^2_0(\Omega)$, which is all we have in Theorem 5.3 even with star-complementation, can still be obtained in this case is unknown; it does not seem likely.

If, by any means, it is possible to prove the exponential decay of solutions of (6.12), (6.13), (6.14) it will lead to a comparable controllability result.

Theorem 6.3. *If for some* $\beta > 0$ *all solutions of* (6.12), (6.13), (6.14) *have uniform exponential decay as* $t \to \infty$, *i.e., if for some* $C, \delta > 0$

$$\|(w(\cdot, t), v(\cdot, t))\|_{\mathcal{H}^1_0(\Omega)} \le C e^{-\delta t} \|(w_0, v_0)\|_{\mathcal{H}^1_0(\Omega)}, \quad t \ge 0, \tag{6.33}$$

then the system (5.25), (5.26), (5.27) *is controllable in* $\mathcal{H}^1_0(\Omega)$ *with controls* $u \in L^2(\Gamma_1 \times [0, T])$ *for some* $T > 0$.

Sketch of Proof. The proof again parallels that of Theorem 2.9 and that of Theorem 5.3. One begins with an arbitrary state $(\hat{w}_0, \hat{v}_0)$ in $\mathcal{H}_0^1(\Omega)$ and supposes that $\hat{w}(x,t)$ is the solution of (6.12), (6.13), (6.14) with this initial state. Taking note of (6.33), we choose T so large that

$$C\,e^{-\delta T} \equiv d < 1 \quad .$$

Then we let

$$\tilde{w}_1 = w_1 - \hat{w}(\cdot, T), \quad \tilde{v}_1 = v_1 - \hat{v}(\cdot, T) \; (= v_1 - \frac{\partial w}{\partial t}(\cdot, T))$$

where $(w_1, v_1) \in \mathcal{H}_0^1(\Omega)$ is the desired terminal state to be achieved at time T. Then we let $\tilde{w}(x,t)$ be the solution of (6.12), (6.13) and

$$\frac{\partial \tilde{w}}{\partial \nu}(x,t) - \beta \frac{\partial \tilde{w}}{\partial t}(x,t) = 0, \quad x \in \Gamma_1, \quad t \leq T \tag{6.34}$$

with

$$\tilde{w}(\cdot, T) = \tilde{w}_1, \; \frac{\partial \tilde{w}}{\partial t}(\cdot, T) = \tilde{\mathbf{v}}_1 \quad .$$

Since time reversal leaves (6.12) and (6.13) invariant and carries (6.14) into (6.34), we have

$$\|\tilde{w}(\cdot, T-t), \; \tilde{v}(\cdot, T-t)\|_{\mathcal{H}_0^1(\Omega)} \leq C\,e^{-\delta t}\,\|(\tilde{w}_1, \; \tilde{v}_1)\|_{\mathcal{H}_0^1(\Omega)} \quad .$$

Thus letting

$$\tilde{w}_0 = \tilde{w}(\cdot, 0), \quad \tilde{v}_0 = \tilde{v}(\cdot, 0)$$

we find that with

$$u(x,t) = \frac{\partial \hat{w}}{\partial \nu}(x,t) + \frac{\partial \tilde{w}}{\partial \nu}(x,t), \quad x \in \Gamma_1, \quad 0 \leq t \leq T \tag{6.35}$$

we have u steering $(\tilde{w}_0 + \hat{w}_0, \tilde{v} + \hat{v}_0)$ to (w_1, v_1) during $[0, T]$. The same sort of contraction fixed point argument as employed in Theorem 5.3 allows us to put

$$\tilde{w}_0 + \hat{w}_0 = w_0, \quad \tilde{v}_0 + \hat{v}_0 = v_0$$

where $(w_0, v_0) \in \mathcal{K}_0^1(\Omega)$ is the given initial state.

Now, how can we see that $u \in L^2(\Gamma_1 \times [0, T])$? We do not get this from the theorem of traces; that would still give $u \in C([0, T]; H^{-\frac{1}{2}}(\Gamma_1))$. What one observes, rather, is that the energy $E(w, v)$ is equivalent to $\|(w, v)\|_{\mathcal{K}_0^1(\Omega)}$, (assuming Γ_0 is not empty) and, e.g. with regard to the solution $\tilde{w}(x, t)$,

$$\begin{aligned} & E(\tilde{w}(\cdot, 0), \tilde{v}(\cdot, 0)) - E(\tilde{w}(\cdot, T), \tilde{v}(\cdot, T)) \\ & = \frac{1}{\beta} \int_0^T \int_\Gamma \left(\frac{\partial \tilde{w}}{\partial \nu}(x, t)\right)^2 ds\, dt , \end{aligned} \tag{6.36}$$

the equality following from the fact that $\tilde{w}$ satisfies (6.34). Since for solutions $\tilde{w}$ in $\mathcal{K}_0^1(\Omega)$ the left hand side of (6.36) must be finite, the right hand side must be also. A similar argument applies to $\hat{w}$ and then (6.35) gives $u \in L^2(\Gamma_1 \times [0, T])$. It should be noted that we have no indication that $u \in C([0, T], L^2(\Gamma_1))$.

With this we conclude the present section, whose purpose has been to demonstrate that some of the relationships between controllability and stabilizability which obtain for finite dimensional systems extend to the higher dimensional wave equation as well. Much remains to be done,

quite clearly, first of all in extending Theorems 5.3 and 6.2 and, eventually, in constructing, if possible, canonical systems as in Section 4 to investigate the spectral assignment properties of the system. There are interesting questions connected with the approximate controllability result of Theorem 5.2 as well. When Γ_1 is just an arbitrary non-empty relatively open subset of $\partial\Omega$ we know (from the rectangle example) that we cannot expect controllability in $\mathcal{H}^1_0(\Omega)$ in general. But what sort of controllability do we have? In particular, is it possible to steer the system state from $(w(\cdot, 0), v(\cdot, 0)) = 0$ to states $(w(\cdot, T), v(\cdot, T)) = (\phi_k, 0)$, $(w(\cdot, T), v(\cdot, T)) = (0, \phi_k)$, where ϕ_k is an eigenfunction of the operator

$$Lw = \frac{1}{\rho(x)} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(\alpha^i_j(x) \frac{\partial w}{\partial x^j} \right)$$

with boundary conditions

$$w(x) = 0, \quad x \in \Gamma_0, \quad \left(\frac{\partial w}{\partial x}(x), A(x)\, v(x)\right)_{E^n} = 0, \quad x \in \Gamma_1 \ ?$$

As yet we do not know. It would also be interesting to discover how this "eigenfunction controllability" is related to spectral assignment problems. When it obtains, is it possible to move individual points of the spectrum via state feedback without affecting other points of the spectrum? These and many other questions wait their turn for treatment.

7. CONTROL THEORY OF PARABOLIC SYSTEMS: RESULTS FOR THE CASE OF ONE SPACE DIMENSION

The control, observation and stabilization theory for the heat equation is very different in character from that of the wave equation and yet, as we shall see, very closely related to it. It has a somewhat longer history than that of the wave equation. In his book [7], A. G. Butkovskiy devoted a great deal of attention to this equation and notes, e.g., application to problems of pre-heating ingots preparatory to rolling operations in the metal working industries. From his work it appears not inaccurate to say that he was primarily interested in optimal control and only secondarily with questions of controllability. A very significant early contribution, from the viewpoint of controllability theory, is due to Egorov [18], [19]. Though primarily interested in the problem of time optimal control, he found it necessary, as part of this study, to characterize a class of reachable states. Another early paper is due to Gal'chuk [30]. Though specialized to the problem of achieving a final equilibrium state, it presents results quite comparable to those which we will describe below.

A great many decisive developments in the controllability theory of the heat equation are due to H. O. Fattorini. These have been presented in numerous articles (see e.g. [20] - [26]) and form a large part of the "core" of the existing theory, including that to be presented here.

We also make reference to recent papers of R. Triggiani ([107], [108]), concerning which we will have more to say in the sequel.

We have seen in Sections 3-5 that the control time T plays a very decisive role in the case of hyperbolic systems in determining whether or not (exact or approximate) controllability obtains. In [20] Fattorini showed that there is no comparable phenomenon associated with control of the heat equation. Indeed it can be shown in many cases that the set of points reachable in time T, starting with the zero initial state, is completely independent of T as long as $T > 0$. It is also possible, in the case of the heat equation, to develop extensions of the rank condition (2.4) for controllability, for which no "realistic" counterpart exists in the case of the wave equation. This question is treated by Triggiani in [108]. We shall briefly examine these questions, and others, in the context of the theory to be developed below.

We will develop our theory on two levels of abstraction. On the one level we shall consider an operator A on a Hilbert space X which generates an analytic semigroup ([39]). To keep the exposition simple we shall suppose A to be an unbounded normal operator on H with compact resolvent $(\lambda I - A)^{-1}$ whose spectrum is confined to a sectorial region

$$|\arg(z - z_0) - \pi| \leq \theta_1 < \pi \ , \tag{7.1}$$

where z_0 is some "base point" in the complex plane. The holomorphic semigroup is then defined for $|\arg z| < \pi - \theta_1$. The control system is

$$\dot{x} = Ax + Bu, \quad x \in X, \quad u \in U \ , \tag{7.2}$$

where U is a second Hilbert space and $B: U \to X$ is a bounded operator. On the second, more concrete level, we shall consider the equation

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} + r(x)w = g(x)\, u(t), \quad t \geq 0, \quad 0 < x < 1 , \tag{7.3}$$

with the function r continuous on $[0,1]$ and $g \in L^2[0,1]$. The boundary conditions are

$$a_0 w(0,t) + b_0 \frac{\partial w}{\partial x}(0,t) = 0, \; a_1 w(1,t) + b_1 \frac{\partial w}{\partial x}(1,t) = 0 , \tag{7.4}$$

$$(a_0)^2 + (b_0)^2 \neq 0, \quad (a_1)^2 + (b_1)^2 \neq 0 .$$

Closely related to this system is the boundary control system

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} + r(x)w = 0 , \tag{7.5}$$

$$a_0 w(0,t) + b_0 \frac{\partial w}{\partial x}(0,t) = 0, \; a_1 w(1,t) + b_1 \frac{\partial w}{\partial x}(1,t) = u(t) . \tag{7.6}$$

The system (7.3), (7.4) is, of course, a special case of (7.2) with $X = L^2[0,1]$, $U = E^1$ and A the Sturm-Liouville operator $Lw = \frac{\partial^2 w}{\partial x^2} - r(x)w$ with the indicated boundary conditions. We shall have more to say concerning specific properties of this operator later.

We begin with (7.2) and the question of approximate controllability. We shall analyze this problem by means of an extension of the rank condition (2.4), following techniques developed by Fattorini [22] and Triggiani [108]. We note first of all, taking the spaces X, Y, Z and the operators C, C^* as in (2.15), (2.19) and $S: X \to Z = X$ as the identity on X, that approximate controllability of (7.2), i.e., the question as to whether the range of C (cf. (2.15)) is dense in X is equivalent to

the distinguishability question (Def. 2.5, part (i)) which in this case amounts to: does

$$y(t) = B^* e^{A^* t} y_0 \equiv 0, \quad t \in [0, T] \tag{7.7}$$

imply

$$y_0 = 0 \; ? \tag{7.8}$$

Since B, and hence B^*, is bounded and since A, and hence A^*, generates a holomorphic semigroup of bounded operators the function $y(t)$ extends to a holomorphic U-valued function $y(z)$ in the sector (cf. (7.1)) $|\arg z| < \pi - \theta_1$. It then admits repeated differentiation to give (note that $(A^*)^k e^{A^* t}$ extends to a bounded operator)

$$B^* (A^*)^k e^{A^* t} y_0 \equiv 0, \quad t \in [0, T], \quad k = 0, 1, 2, \ldots \tag{7.9}$$

So the above question may be replaced by the question as to whether or not (7.9) implies (7.8).

It is really not possible to discuss here all of the situations in which it has been established that (7.9) implies $y_0 = 0$, so that (7.2) is approximately controllable. For the case where A is an unbounded self-adjoint operator on the Hilbert space X this analysis was carried out by Fattorini in [20], making use of the notion of the ordered representation of such an operator. In [20] a more general result, where A is the generator of a semigroup in a Banach space X, is obtained in terms of the spectral sets of the operator A. A great variety of theorems of this sort are discussed and presented by Triggiani in [108].

In order to give some notion as to the manner in which results of this type are obtained we shall consider first the case where, in place of our earlier assumptions, A is taken to be a compact normal operator on the Hilbert space X. We shall further suppose that if zero is an eigenvalue of A then the associated eigenspace is finite dimensional. Let the eigenvalues of A be $\lambda_1, \lambda_2, \lambda_3, \ldots$ and let the associated mutually orthogonal eigenspaces in X, be $X_1, X_2, X_3, \ldots$ each spanned by the orthonormal set of eigenvectors $\phi_{i,1}, \phi_{i,2}, \ldots, \phi_{i,n_i}$, where n_i is the dimension of X_i. Then each element $y_0 \in X$ has the unique expansion

$$y_0 = \sum_{i=1}^{\infty} \sum_{j=1}^{n_i} \eta_{i,j} \phi_{i,j} .$$

Let $u_1, u_2, u_3, \ldots$ be a basis for the control space U. For each i we form the array

$$\begin{matrix} (u_1, B^* \phi_{i,1}) & (u_1, B^* \phi_{i,2}) & \cdots & (u_1, B^* \phi_{i,n_i}) \\ (u_2, B^* \phi_{i,1}) & (u_2, B^* \phi_{i,2}) & \cdots & (u_2, B^* \phi_{i,n_i}) \\ \vdots & \vdots & & \vdots \\ (u_k, B^* \phi_{i,1}) & (u_k, B^* \phi_{i,2}) & \cdots & (u_k, B^* \phi_{i,n_i}) \\ \vdots & \vdots & & \vdots \end{matrix} \tag{7.10}$$

Then we have the proposition (adapted from [108]).

Proposition 7.1. Assuming A to be compact, as above, the control system (7.2) is approximately controllable for each $T > 0$ if and only if any one of the following three equivalent conditions are satisfied:

(i) (7.9) implies (7.8);

(ii) $N = \text{Range } B + \text{Range } AB + \dots + \text{Range } A^k B + \dots = X$.

(iii) for each $i = 1, 2, 3, \dots$ the array (7.10) has n_i linearly independent rows.

Proof. Assume some array (7.10) has fewer than n_i independent rows. Then there are $\alpha_1, \alpha_2, \dots, \alpha_{n_i}$ not all zero such that with $y_0 = \alpha_1 \phi_{i,1} + \alpha_2 \phi_{i,2} + \dots + \alpha_{n_i} \phi_{i,n_i}$,

$$(u_k, B^* y_0) = 0, \quad k = 1, 2, \dots$$

which, since the u_k form a basis for U, implies

$$B^* y = 0 . \tag{7.11}$$

Here $e^{A^* t} y_0 = e^{\bar{\lambda}_i t} y_0$ so (7.11) gives

$$B^* e^{A^* t} y_0 = 0, \quad k = 1, 2, \dots, \quad t \geq 0 .$$

Since $y_0 \neq 0$, (7.8) does not hold and we do not have approximate controllability.

Suppose on the other hand the array (7.11) does contain n_i linearly independent rows for each i. Now, since A is normal and N is clearly invariant under A, $N^\perp$ is also invariant under A and, if not $= \{0\}$, contains an eigenvector, ϕ, for A – which then must be one of the $\phi_{i,k}$ described earlier. But then

$$B^* \phi = 0$$

since $N^\perp$ is included in the null space of B^*. But $B^* \boldsymbol{\phi} = 0$ implies that one of the arrays (7.11) cannot have n_i linearly independent rows – contrary to our hypothesis. We conclude $N^\perp = \{0\}$. But $N^\perp$ consists of y_0 for which $B^*(A^*)^k y_0 = 0$, $k = 0, 1, 2, \ldots$ so we conclude the implication (7.9) ⇒ (7.8), and hence (7.7) ⇒ (7.8), is true, giving approximate controllability in this special case. The complete demonstration of the equivalence of (i), (ii), (iii) is a standard exercise.

Naturally one wishes to extend these arguments to the case of the original generator A whose resolvent was assumed compact. This is discussed by Triggiani in [108] and by Fattorini in [21], [22]. In [20] Fattorini shows that approximate controllability of (7.2) is equivalent to approximate controllability of

$$\frac{dx}{dt} = (\lambda I - A)^{-1} x + Bu, \quad \lambda \in \rho(A) .$$

Assuming this result, Proposition 7.1 lends to

Theorem 7.2. Let the resolvent $(\lambda I - A)^{-1}$ of A be compact. Then approximate controllability of (7.2) for any $T > 0$ is equivalent to the condition that the array (7.11) should have maximal rank for each $i = 1, 2, 3, \ldots$.

In general these results do not extend to the wave equation because the semigroup (group, actually, in that case) is not holomorphic and (7.8) is not equivalent to (7.7), in fact $(A^*)^k e^{A^* t}$ may not even

be defined on the whole space X - as it is in the holomorphic case. A very curious paradox arises here, however, and it is discussed in [108]. Consider the wave equation

$$\frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial x^2} + r(x)w = g(x)\, u(t) \quad , \qquad (7.12)$$

with boundary conditions (7.1). Let the Sturm-Liouville operator $-\frac{\partial^2 w}{\partial x^2} - r(x)w$, with the related homogeneous boundary conditions, have eigenvalues $-\lambda_1, -\lambda_2, -\lambda_3, \ldots$ and associated orthonormal eigenvectors $\phi_1, \phi_2, \phi_3, \ldots$ forming a basis for $L^2[0,1]$. Then we have the expansion

$$g = \sum_{k=1}^{\infty} g_k \phi_k \; .$$

We shall suppose no $g_k = 0$ and, in addition, that the g_k decrease in magnitude so rapidly that g is an "analytic vector", which means that

$$\sum_{k=1}^{\infty} e^{\pm i(\lambda_k)^{\frac{1}{2}} t} g_k \, y_{0,k}$$

is holomorphic in some set which includes the real axis whenever $\sum_{k=1}^{\infty} (y_{0,k})^2 < \infty$. In this case the degree of controllability from the standpoint of Section 3-5 is very weak but, curiously enough, for the semigroup e^{At} associated with (7.3), (7.4) the conditions (7.9) and (7.7) continue to be equivalent - not as a consequence of any analyticity of e^{At}, but as a consequence of the rapid decrease in the magnitudes of the g_k. As a result (7.3), (7.4) is approximately controllable for every $T > 0$ for such "analytic vectors" g - even though the results

parallel to those of Section 3-5 ([87]) apply only for $T \geq 2$. Here, for $T < 2$, we have approximate controllability without "eigenfunction controllability" (see end of previous section). Eigenfunction controllability is equivalent to the existence of biorthogonal functions for the $e^{\pm i(\lambda_k)^{\frac{1}{2}} t}$ (corresponding to the $e^{\sigma_k t}$ of Section 4) and such biorthogonal functions do not exist for $T < 2$. In addition to being of interest in its own right, this example may indicate why we have asked the question concerning eigenfunction controllability at the end of Section 6.

In the period 1968-1971, roughly speaking a number of independent attempts were made to obtain a more detailed understanding of the control and observation theory of the heat equation on a spatial region consisting of an interval in R^1 . It is interesting that the control and observation studies were carried out independently with the researchers involved not fully realizing the complete equivalence of these problems. It is also true that the results obtained were, in some sense, only an " ε-improvement" over those obtained earlier by Egorov and Gal'chuk ([18], [9], [30]) though a reading of the various papers shows that a more precise understanding of the essential nature of the control and related observation problems was developing.

For our expository treatment it will be sufficient to consider the controlled heat, or diffusion, process (7.3), (7.4) with the same hypotheses on r, g, a_0, b_0, a_1, b_1 as set forth previously. As anticipated in earlier paragraphs, we let $-\lambda_k$, $\phi_k(x)$ denote the eigenvalues and

associated eigenfunctions for the operator $\frac{\partial^2 w}{\partial x^2} - r(x)w$ with boundary conditions of the form (7.4).

The existence and uniqueness theory for (7.3), (7.4) with initial state

$$w(x, 0) = w_0(x), \quad w_0 \in L^2[0,1] , \tag{7.13}$$

and expansion

$$w_0 = \sum_{k=1}^{\infty} w_{0,k} \phi_k \tag{7.14}$$

is summarized in [27] and more details appear in [29] and other standard references on partial differential equations. Suffice it to say that the solution $w = w(x,t)$ of (7.3), (7.4), (7.13) can be expanded as series in the eigenfunctions $\phi_k(x)$, convergent in $L^2[0,1]$:

$$w(\cdot, t) = \sum_{k=1}^{\infty} w_k(t) \phi_k ,$$

and the $w_k(t)$, $k = 1, 2, 3, \ldots$ can be seen to satisfy the first order ordinary differential equations

$$\frac{dw_k}{dt} + \lambda_k w_k = g_k u(t), \qquad k = 1, 2, 3, \ldots \tag{7.15}$$

and initial conditions

$$w_k(0) = w_{0,k} , \tag{7.16}$$

where the $w_{0,k}$ are specified by (7.14) and the g_k are the coefficients in the expansion of g:

$$g = \sum_{k=1}^{\infty} g_k \phi_k \text{ in } L^2[0,1] .$$

From (7.15), (7.16) we have, for $T > 0$,

$$w_k(T) = e^{-\lambda_k T} w_{0,k} + \int_0^T e^{-\lambda_k (T-t)} g_k u(t)dt .$$

For the moment let us consider the null controllability problem where we want $w_k(T) = 0$, $k = 1, 2, 3, \ldots$. What we want then is (cf. Section 2)

$$\mathcal{R}(S) \subseteq \mathcal{R}(C) \tag{7.17}$$

where $S: X \to Z$, $C: Y \to Z$, $X = Z = L^2[0,1]$, $Y = L^2[0,T]$ are defined by

$$S(\sum_{k=1}^{\infty} w_{0,k} \phi_k) = \sum_{k=1}^{\infty} e^{-\lambda_k T} w_{0,k} \phi_k , \tag{7.18}$$

$$Cu = \sum_{k=1}^{\infty} (\int_0^T e^{-\lambda_k (T-t)} g_k u(t)dt)\phi_k . \tag{7.19}$$

The dual observation problem involves the homogeneous equation

$$\frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial x^2} + r(x)y = 0 \tag{7.20}$$

with boundary conditions

$$a_0 y(0,t) + b_0 \frac{\partial y}{\partial x}(0,t) = a_1 y(1,t) + b_1 \frac{\partial y}{\partial x}(1,t) = 0 \tag{7.21}$$

and observation

$$\omega(t) = (y(\cdot, t), g)_{L^2[0,1]} . \tag{7.22}$$

For continuous (final) state observability it is required (Def. 2.5) that for some $K > 0$

$$\|\omega\|_{L^2[0,T]} \geq K\|y(\cdot, T)\|_{L^2[0,1]}$$

for each solution y of (7.20), (7.21) with

$$y(\cdot, 0) = \sum_{k=1}^{\infty} y_{0,k} \phi_k \quad \text{in } L^2[0,1] . \tag{7.23}$$

Using integration by parts one readily sees that for solutions w, y of (7.3), (7.4), (7.20), (7.21) respectively, we have

$$(y(\cdot 0), w(\cdot, T))_{L^2[0,1]} - (y(\cdot, T), w(\cdot, 0))_{L^2[0,1]}$$

$$= \int_0^T (y(T-t), gu(t))_{L^2[0,1]} dt . \tag{7.24}$$

With $u(t) \equiv 0$ and (from (7.18)) the observation that, in this case $w(\cdot, T) = S w(\cdot, 0)$, we infer

$$S^* y(\cdot, 0) = y(\cdot, T)$$

and with $w(\cdot, 0) = 0$ we have $w(\cdot, T) = Cu$ so (7.23) gives

$$(y(\cdot, 0), Cu)_{L^2[0,T]} = \int_0^T (y(T-t), g)_{L^2[0,T]} u(t)dt$$

$$= (\text{cf. } (7.22)) = (\omega(T - \cdot), u)_{L^2[0,T]}$$

and thus

$$C^* y(\cdot, 0) = \omega(T - \cdot) .$$

The observation problem equivalent to null controllability is to determine whether, for some $K > 0$ we have

$$\|\omega(T - \cdot)\|_{L^2[0,T]} = \|C^* y(\cdot, 0)\|_{L^2[0,T]} \geq K \|S^* y(\cdot, 0)\|_{L^2[0,1]} = K\|y(\cdot, T\|_{L^2[0,1]} \tag{7.25}$$

In [27] Fattorini and the present author studied the problem in the control context while Mizel and Seidman [67], [68], and later Dolecki [14], studied the problem in the observation form (7.25).

Just as in Section 4 the problem reduces in the end to a question of linear independence of exponential functions in $L^2[0, T]$, but now it is the functions $e^{-\lambda_k t}$, $k = 1, 2, 3, \ldots$ rather than $e^{\sigma_k t}$, $-\infty < k < \infty$, that we are concerned with. This is an old problem with roots in the theory of Dirichlet series. In [45] Kaczmarz and Steinhaus discuss the Muntz-Szasz problem of completeness of sequences $\{e^{-\mu_k t}\}$ in $L^2[0, T]$ (actually they take $T = 1$ but this is unimportant). The classical result is that $\{e^{-\mu_k t} \mid k = 1, 2, 3, \ldots\}$ are complete in $L^2[0, T]$ (but are not independent) if

$$\sum_{k=1}^{\infty} \frac{1}{\mu_k} = \infty$$

while if

$$\sum_{k=1}^{\infty} \frac{1}{\mu_k} < \infty$$

the $e^{-\mu_k t}$ are independent but span a proper closed subspace of $L^2[0,T]$. Since $\lambda_k = k^2 + \mathcal{O}(1)$ we have $\sum_{k=1}^{\infty} \frac{1}{\lambda_k} < \infty$ and it is the second situation which applies to our case. In this case it has been known for some time [45], [99] that no function $e^{-\lambda_k t}$ lies in the $L^2[0, T]$ closure of the remaining functions $e^{-\lambda_\ell t}$, $\ell \neq k$. We refer to this as "strong" independence. It is clearly equivalent to the existence of biorthogonal functions.

Let $E(\Lambda, T)$ denote the closed subspace of $L^2[0,T]$ spanned by $\{e^{-\lambda_k t} \mid k = 1, 2, 3, \ldots\}$ and let $E(\Lambda, k, T)$ denoted the closed subspace spanned by $\{e^{-\lambda_\ell t} \mid \ell \neq k\}$. For each k let r_k be the closest point to $e^{-\lambda_k t}$ in $E(\Lambda, k, T)$:

$$\min_{r \in E(\Lambda, k, T)} \|e^{-\lambda_k \cdot} - r\|_{L^2[0,T]} = \|e^{-\lambda_k \cdot} - r_k\|_{L^2[0,T]} .$$

Elementary Hilbert space theory shows that with

$$p_k(t) = \frac{e^{-\lambda_k t} - r_k(t)}{\|e^{-\lambda_k \cdot} - r_k\|^2} \quad t \in [0, T]$$

we have

$$(p_k, e^{-\lambda_\ell \cdot})_{L^2[0,T]} = 0, \quad \ell \neq k \tag{7.26}$$

$$(p_k, e^{-\lambda_k \cdot})_{L^2[0,T]} = (p_k, e^{-\lambda_k \cdot} - r_k)_{L^2[0,T]} = 1 \tag{7.27}$$

so that the p_k form a biorthogonal set in $E(\Lambda, T)$ (with the inner product induced by $L^2[0,T]$) to the functions $e^{-\lambda_k \cdot}$.

To study the question as to whether or not (7.25) is true, we note that

$$\|S^* y(\cdot, 0)\|^2_{L^2[0,1]} = \|y(\cdot, T)\|^2_{L^2[0,1]} = \sum_{k=1}^{\infty} |e^{-\lambda_k T} y_{0,k}|^2 . \tag{7.28}$$

Now, using (7.23), (7.25), (7.26), (7.27) and assuming $g_k \neq 0$, $k = 1, 2, 3 \ldots$

$$e^{-\lambda_k T} y_{0,k} = \frac{e^{-\lambda_k T}}{g_k} g_k y_{0,k}$$

$$= \frac{e^{-\lambda_k T}}{g_k} \int_0^T p_k(t) \left(\sum_{\ell=1}^{\infty} g_\ell e^{-\lambda_\ell t} y_{0,\ell}\right) dt$$

$$= \frac{e^{-\lambda_k T}}{g_k} (p_k, \omega)_{L^2[0,T]} = \frac{e^{-\lambda_k T}}{g_k} (p_k(T, \cdot), C^* y(\cdot, 0))_{L^2[0,T]}$$

$$\leq \frac{e^{-\lambda_k T}}{|g_k|} \|p_k\|_{L^2[0,T]} \|C^* y(\cdot, 0)\|_{L^2[0,T]}$$

and then (7.28) gives

$$\|S^* y(\cdot, 0)\|^2_{L^2[0,L]} \leq \sum_{k=1}^{\infty} \frac{e^{-2\lambda_k T}}{|g_k|^2} \|p_k\|^2_{L^2[0,T]} \|C^* y(\cdot, 0)\|^2_{L^2[0,T]} .$$

Hence, if one can establish that there is some $M > 0$ such that

$$\sum_{k=1}^{\infty} \frac{e^{-2\lambda_k T}}{|g_k|^2} \|p_k\|^2_{L^2[0,T]} \leq M^2 \tag{7.29}$$

then $K = \frac{1}{M}$ is such that (7.25) holds and we have both observability in that sense and null controllability. We will establish (7.29) under the basic assumption (usually satisfied in practice) that, for some $\gamma > 0$ and integer $m > 0$,

$$|g_k| \geq \gamma k^{-m}, \qquad k = 1, 2, 3, \ldots \tag{7.30}$$

To establish (7.29) it is clear that one must have bounds on the norms of the functions p_k . This has been done repeatedly in the

literature of the past ten years; usually, but not always ([62]) in the context of control or observation for the heat equation.

In 1962, in the course of studying the time optimal control problem for the heat equation, Yu. V. Egorov obtained in [18], [19] a controllability result. As stated in [18] it appears to be restricted to the case where $r(x) \equiv 0$ in (7.3). It is, in that context, equivalent to showing that

$$\|p_k\| \leq K_0 \exp(K_1 \omega_k^{1+\varepsilon}) \; (\omega_k = \sqrt{\lambda_k}\;) \tag{7.31}$$

for an arbitrary $\varepsilon > 0$; K_0, K_1 positive constants.

In 1969 Mizel and Seidman published a paper [67] on the observation problem for the heat equation on a finite interval, obtaining a result equivalent to

$$\|p_k\| \leq K_0 \exp(K_1 \lambda_k) \; . \tag{7.32}$$

(From their references it seems probable that they were not aware of Egorov's result.)

If (7.32) is used, then (7.29) holds for $T > K_1/2$, indicating that the control and observation problems are solvable for such T and, insofar as observation is concerned, this is the result of [67]. Mizel and Seidman were skeptical that this really was the best result. If (7.31) is used it becomes clear that (7.29) holds for arbitrary $T > 0$. Mizel and Seidman returned in 1972 with a second paper [68] and, making use of a result of L. Schwartz [99] which implies that the natural restriction map from

$E(\Lambda, T_2)$ to $E(\Lambda, T_1)$ $(T_2 > T_1)$ is boundedly invertible, removed the restriction on T, showing the observation problem to be solvable for arbitrary $T > 0$. In fact this result also follows from the work of Fattorini referred to earlier, whose theorem on the time invariance of the reachable set is closely related to Schwartz' result on the isomorphism of $E(\Lambda, T_1)$ and $E(\Lambda, T_2)$. We shall have more to say on this.

In 1971 Fattorini and the present author collaborated in an attack on the controllability problem for the system (7.3), (7.4), aware of Egorov's result [18] but not of the cited work of Mizel and Seidman. The result was a paper [27] in which the control problem was solved in two different ways – but with essentially the same result both times. One method involved the estimation of the norm of the biorthogonal functions p_k introduced above. Following the results in [45] and [99] which imply that

$$\|p_k\|_{L^2[0,T]} \le \frac{\lambda_k K_T}{2} \prod_{j=1}^{\infty} \left(1 + \frac{\lambda_k}{\lambda_j}\right) \Big/ \prod_{\substack{j=1 \\ j \neq k}}^{\infty} \left(1 - \frac{\lambda_k}{\lambda_j}\right),$$

it was found by very careful estimation that for $K_0, K_1 > 0$

$$\|p_u\|_{L^2[0,T]} \le K_0 \exp(K_1 \omega_k), \quad \omega_k = \sqrt{\lambda_k} \qquad (7.33)$$
$$k = 1, 2, 3, \ldots$$

and it was seen that this is the best possible result. This improves Egorov's result (7.31) by "ε" and again shows that (7.29) holds for arbitrary $T > 0$. A second approach refined results of R. M. Redheffer [83],

using the Fourier transform to obtain biorthogonal functions q_k, not lying in $E(\Lambda, T)$ but still in $L^2[0,T]$, such that $(q_\ell, e^{-\lambda_k \cdot})_{L^2[0,T]} = \delta_\ell^k$, and an estimate, for $\hat{K}_0, \hat{K}_1 > 0$

$$\|q_k\|_{L^2[0,T]} \leq \hat{K}_0 \exp(\hat{K}_1 \omega_k), \quad w_k = \sqrt{\lambda_k} \tag{7.34}$$

$$k = 1, 2, 3, \ldots ,$$

which yields the same results. Since, necessarily,

$$\|p_k\|_{L^2[0,T]} \leq \|q_k\|_{L^2[0,T]}$$

the result (7.34) implies the result (7.33) but the first approach allows explicit identification of K_1 while the second does not. This "Fourier transform" method would be more or less of a curiosity were it not for the fact that, with slight modification, described in the next section, it allows one to see that each controllability result obtained for the wave equation yields a corresponding result for the heat equation.

We have noted that with S and C defined as above, the observability result (7.25) is equivalent to null controllability, i.e., that for each initial state $w_0 \in L^2[0,1]$ there is a control $u \in L^2[0,T]$ such that the resulting solution of (7.3), (7.4) satisfies

$$w(\cdot, T) = 0 .$$

Each of the above results bounding $\|p_k\|_{L^2[0,T]}$ can, however, be applied to identification of terminal states $w(\cdot, T) = w_T$ other than zero which can be achieved using controls in $L^2[0,T]$. For this part of the

problem we may without loss of generality take $w(\cdot, 0) = 0$. From (7.15) then, if

$$w_T = \sum_{k=1}^{\infty} w_{T,k} \phi_k , \tag{7.35}$$

we have

$$w_{T,k} = \int_0^T e^{-\lambda_k (T-t)} g_k u(t)dt . \tag{7.36}$$

Writing $u(t)$ as a formal series in the p_k :

$$u(t) = \sum_{k=1}^{\infty} \mu_k p_k(T-t)$$

the biorthogonality property (7.26), (7.27) of the p_k relative to the $e^{-\lambda_k t}$ gives

$$w_{T,k} = g_k \mu_k \quad \text{or} \quad \mu_k = \frac{w_{T,k}}{g_k} .$$

For the given final state (7.35) we then have the control

$$u(t) = \sum_{k=1}^{\infty} \frac{w_{T,k}}{g_k} p_k(T-t) \tag{7.37}$$

and $w(\cdot, T)$ is reachable if (and only if) the series (7.37) converges in $L^2[0,T]$. From (7.30) and (7.33) we see that this is certainly true if

$$|w_{T,k}| \le A_0 \exp(-A_1 \omega_k), \qquad k = 1, 2, 3, \ldots , \tag{7.38}$$

for $A_0 > 0$ and $A_1 > K_1$, which in turn is true if the desired terminal state is in the domain of the unbounded operator $\exp(A_1(-L)^{\frac{1}{2}})$ (see (7.6) ff.).

We summarize the results developed above in

Theorem 7.3. Consider the control system (7.3), (7.4) with the control distribution function g satisfying (7.30). Then, given any positive number T, an initial state

$$w_0 = \sum_{k=1}^{\infty} w_{0,k} \phi_k \in L^2[0,1] \tag{7.39}$$

and a terminal state

$$w_T = \sum_{k=1}^{\infty} w_{T,k} \phi_k , \tag{7.40}$$

such that w_T belongs to the domain of the unbounded operator $\exp(A_1(-L)^{\frac{1}{2}})$, there is a control $u \in L^2[0,T]$ steering a solution of (7.3), (7.4) from w_0 to w_T during $[0,T]$. Moreover, the positive constant A_1 is independent of T.

For definiteness we stipulate that $(-L)^{\frac{1}{2}}$ is defined in such a way that its spectrum $\{\omega_k | w_k^2 = \lambda_k, \quad k = 1, 2, 3, \dots\}$ lies in $\{0\} \cup \{\lambda | -\frac{\pi}{2} < \arg(\lambda) \leq \frac{\pi}{2}\}$.

We remark that it is known ([27]) that this is the best result which can be obtained if we insist that (7.37) converge "absolutely", i.e.,

$$\sum_{k=1}^{\infty} \left| \frac{w_{T,k}}{g_k} \right| \| p_k \|_{L^2[0,T]} < \infty .$$

But Theorem 7.3 does not characterize all reachable finite states. For $L = \pi$, $r(x) \equiv 0$, $b_0 = b_1 = 0$, $g_k = \frac{1}{k}$ we have $\lambda_k = k^2$ and the initial state $w_0 = 0$ and control $u(t) \equiv 1$ yields the final state

$$w_{T,k} = \frac{1}{k} \int_0^T e^{-k^2(T-t)}\, dt$$

$$= \frac{1}{k^3} [1 - e^{-k^2 T}]$$

for which (7.38) is not even "nearly" satisfied.

The problem of boundary control of the heat equation, wherein (7.3), (7.4) is replaced by (7.5), (7.6) is shown in [27] to be reducible to the form studied above, with the g_k bounded and bounded away from zero if $b_1 \neq 0$ and growing like k when $b_1 = 0$. No separate treatment is required.

We continue this section by proving, for the special case of the system (7.3), (7.4) considered above, the theorem on the time invariance of the reachable set, referred to briefly at the beginning of the section. We use the notation of (7.17) ff. with the modification that, since the whole class of intervals $[0, T]$, $T > 0$, will be considered, the control to state operator will be supplied with a subscript, viz., C_T, to indicate the control interval under discussion. The question of controllability concerns, of course, the range of C_T. If $T_2 > T_1$ then clearly, since we can take $u(t) \equiv 0$ for $0 \leq t < T_2 - T_1$,

$$\mathcal{R}(C_{T_2}) \supseteq \mathcal{R}(C_{T_1}) . \tag{7.41}$$

From the duality theory, this is equivalent to

$$\| C_{T_2}^* y \| \geq M_1 \| C_{T_1}^* y \|, \quad y \in Y , \tag{7.42}$$

for some $M_1 > 0$.

We have invariance of the reachable set (from the zero initial state) provided the inclusion in (7.42) can be reversed, which is equivalent to reversing the inequality in (7.42).

Now for the system (7.3) we have

$$\omega_T(t) = (C_T^* w)(t) = (w(\cdot, t), g)_{L^2[0,1]} = \sum_{k=1}^{\infty} \bar{g}_k \, w_k(t)$$

$$= \sum_{k=1}^{\infty} \bar{g}_k \, w_{0,k} \, e^{-\lambda_k t}, \quad t \in [0, T] \,, \tag{7.43}$$

for any initial state $w_0 = \sum_{k=1}^{\infty} w_{0,k} \, \phi_k$. The inequality (7.42) is the same as

$$\| \omega_{T_2} \|_{L^2[0, T_2]} \geq M_1 \| \omega_{T_1} \|_{L^2[0, T_1]} \,.$$

If we can establish that for some $K_2 > 0$, independent of w_0, we have

$$\| \omega_{T_1} \|_{L^2[0, T_1]} > M_2 \| \omega_{T_2} \|_{L^2[0, T_2]} \tag{7.44}$$

we have our result.

To this end, suppose we have a sequence $\{\omega^\ell\}$ corresponding to initial states ω_0^ℓ with

$$\| \omega_{T_2}^\ell \|_{L^2[0, T_2]} = 1, \quad \ell = 1, 2, 3, \ldots \tag{7.45}$$

while

$$\lim_{\ell \to \infty} = \| \omega_{T_1}^\ell \|_{L^2[0, T]} = 0 \,. \tag{7.46}$$

Let $\{p_k\}$ be the sequence of elements biorthogonal to the $e^{-\lambda_k t}$ in $L^2[0, T_2]$, as introduced prior to Theorem 7.3. Thus

$$(e^{-\lambda_k \cdot}, p_j) = \delta_j^k .$$

Using (7.33), (7.43) we have

$$|\bar{g}_k w_{0,k}^{\ell}| = |\int_0^{T_2} \omega^{\ell}(t)\, p_k(t)dt|$$

$$\leq K_0 \exp(K_1 \lambda_k^{\frac{1}{2}}) \|\omega_{T_2}^{\ell}\|_{L^2[0,T_2]}$$

from which it follows from (7.43) that

$$|\omega^{\ell}(z)| \leq M_{\mu}, \quad Re(z) \geq \mu$$

for any positive μ. Applying Montel's theorem [38] we conclude that a subsequence of $\{\omega^{\ell}\}$ which we will still call $\{\omega^{\ell}\}$, converges uniformly to an analytic function $\tilde{\omega}(z)$ in compact subset of $Re(z) \geq \mu$. Taking $\mu < T_1$ and using (7.46) we conclude that $\tilde{\omega}(z) = 0$. Then (7.46) combined with the uniform convergence on $[\mu, T_2]$ gives

$$\lim_{\ell \to \infty} \|\omega_{T_2}^{\ell}\|_{L^2[0,T_2]} = 0 ,$$

which contradicts (7.45). We conclude (7.46) is impossible, which implies (7.44) for some $M_2 > 0$, and we have the desired result.

We remark that the above argument is essentially the same as that of Schwartz, referred to in the paragraph following (7.32). The result has been extended recently by Fattorini [26] to cover the problems of

boundary control of higher dimensional parabolic systems which we will discuss in Section 8.

Other work closely related to what we have described so far in this section has been done by S. Dolecki [14]. He is concerned with continuous final state observability of (7.20), (7.21) in the case where the observation (7.22) is modified to

$$\omega(t) = y(\hat{x}, t) \quad ,$$

where $\hat{x}$ is a point in the interval $[0,1]$. Making use of number theoretic properties of $\hat{x}$ he establishes that continuous final state observability holds for almost all $\hat{x} \in [0,1]$ but there is a dense subset for which such continuous final state observability does not hold. He is also concerned with the case wherein $\hat{x} = \hat{x}(t)$ varies with time, t.

If the function $r(x)$ in (7.3) is positive and in (7.4) either a_0 or a_1 is different from zero all eigenvalues $-\lambda_k$ of the Sturm-Liouville operator $-Lw = \frac{\partial^2 w}{\partial x^2} - r(x)w$ are negative and the uncontrolled system ((7.3), (7.4) with $u(t) \equiv 0$) is asymptotically stable. Under other circumstances, however, it is possible that finitely many of the $-\lambda_k$, say for $k = 1, 2, 3, \ldots, \hat{K}$, are non-negative and some sort of stabilizing control is called for. Even if all $-\lambda_k$ are negative it may be desired to move some of them further to the left in order to improve the "settling time" of the system. This type of problem has been considered by Triggiani in [107] and also by Sakawa [98]. Letting P and Q be the

orthogonal projections onto the subspaces of $L^2[0,1]$ spanned by $\lambda_1, \lambda_2, \ldots, \lambda_{\hat{K}}$ and $\lambda_{\hat{K}+1}, \lambda_{\hat{K}+2}, \ldots,$ respectively, and now reverting to the abstract system (7.2), we obtain

$$P\dot{x} = PAx + PBu = APx + PBu \ ,$$

$$Q\dot{x} = QAx + QBu = AQx + QBu \ ,$$

which, with an obvious notation, we can represent by

$$\begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u \ . \tag{7.47}$$

If this system is to be even approximately controllable the system

$$\dot{x}^1 = A_1 x^1 + B_1 u$$

(which, at least insofar as x^1 is concerned, is finite dimensional) must be controllable. One can easily see then that there is a linear feedback law

$$u = Kx^1 \tag{7.48}$$

such that

$$\dot{x}^1 = (A_1 + B_1 K)x^1$$

has any desired eigenvalues $\mu_1, \mu_2, \ldots, \mu_K$ in place of the eigenvalues $-\lambda_1, -\lambda_2, \ldots, -\lambda_{\hat{K}}$. The system obtained by substituting (7.48) into (7.47), viz.,

$$\begin{pmatrix} \dot{x}^1 \\ \dot{x}^2 \end{pmatrix} = \left(\begin{array}{c|c} A_1 + B_1 K & 0 \\ B_2 K & A_2 \end{array}\right) \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

clearly has eigenvalues $\mu_1, \mu_2, \ldots, \mu_{\hat{K}}, -\lambda_{\hat{K}+1}, -\lambda_{\hat{K}+2}, \ldots$ and is asymptotically stable if the μ_i, $i = 1, 2, \ldots, K$ have negative real parts.

The above result, which shows that finitely many eigenvalues can be moved at will without moving the others, can be strengthened by a variant of the methods used in Section 4. In [96] it is shown that for the system (7.3), (7.4) one can achieve any desired eigenvalues $\mu_1, \mu_2, \mu_3, \ldots$ with (cf. (4.55))

$$\sum_{j=1}^{\infty} \left| \frac{\lambda_j - \mu_j}{j g_j} \right|^2 < \infty$$

by use of feedback

$$u(t) = (w(\cdot, t), k)_{L^2[0,1]}$$

in (7.3), (7.4), $k \in L^2[0,1]$ being appropriately chosen.

8. HIGHER DIMENSIONAL PARABOLIC SYSTEMS; HARMONIC ANALYSIS OF BOUNDARY VALUES OF SOLUTIONS

Our next objective is to study the heat equation which "corresponds" to the wave equation (5.1) introduced in Section 5:

$$\rho(x)\frac{\partial w}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i}\left(\alpha_j^i(x)\frac{\partial w}{\partial x^j}\right) = 0, \quad t \geq 0, \quad x \in \Omega , \tag{8.1}$$

with ρ, α_j^i, Ω having the same properties as in that section. In particular

$$\partial\Omega \equiv \Gamma = \Gamma_0 \cup \Gamma_1$$

and we suppose control to be exercised, through the boundary conditions, only on the portion Γ_1 of the boundary. For simplicity, and to achieve correspondence with our work in Section 5, we shall suppose the boundary conditions are

$$w(x,t) = 0, \quad x \in \Gamma_0, \ t \geq 0 , \tag{8.2}$$

$$\left(\frac{\partial w}{\partial x}(x,t),\ A(x)\,\nu(x)\right)_{E^n} = f(x,t), \quad x \in \Gamma_1 \quad t \geq 0 , \tag{8.3}$$

f, required to lie in $L^2(\Gamma_1 \times [0,\tau])$ for any $\tau > 0$, being our control function. An existence and regularity result for (8.1), (8.2), (8.3) is referenced in [28].

The controllability – observability duality relationship is important here, as always. Letting $z = z(x,t)$ satisfy equations of the form (8.1), (8.2), (8.3) but with $f \equiv 0$ we have, for any $\tau > 0$, $0 < t < \tau$,

$$\frac{d}{dt} \int_\Omega w(x,t)\, z(x,\tau-t)\, \rho(x)dx$$

$$= \int_\Omega \left[\sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} (\alpha_j^i(x) \frac{\partial w}{\partial x^j}(x,t) \right] z(x,\tau-t) - w(x,t) \left[\sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} (\alpha_j^i(x) \frac{\partial z}{\partial x^i}(x,\tau-t \right] dx$$

$$= \int_\Omega \operatorname{div}_x \left[(A(x) \frac{\partial w}{\partial x}(x,t))\, z(x,\tau-t) - w(x,t)\, (A(x) \frac{\partial z}{\partial x}(x,\tau-t) \right] dx$$

$$\int_{\partial\Omega} \left[(\frac{\partial w}{\partial x}(x,t),\; A(x)\,\nu(x))\, z(x,\tau-t) - w(x,t)\, (\frac{\partial z}{\partial x}(x,t),\; A(x)\,\nu(x)) \right] ds$$

$$= \int_{\Gamma_1} f(x,t)\, z(x,\tau-t) ds \tag{8.4}$$

where s denotes surface measure on $\partial\Omega$. The integrated form of this is, for $\tau > 0$,

$$\int_\Omega w(x,\tau)\, z(x,0)\, \rho(x)dx - \int_\Omega w(x,0)\, z(x,\tau)\, \rho(x)dx =$$

$$\int_0^\tau \int_\Omega f(x,t)\, z(x,\tau-t) ds\, dt \quad .$$

Following the schema of Section 2, we let $X = Z = L_\rho^2(\Omega)$, which is just $L^2(\Omega)$ with the "weighted" inner product

$$(w, \hat{w})_{L_\rho^2(\Omega)} = \int_\Omega w(x)\, \hat{w}(x)\, \rho(x)dx$$

and we let $Y = L^2(\Gamma_1 \times [0,\tau])$. We define $S: L_\rho^2(\Omega) \to L_\rho^2(\Omega)$ by

$$S w_0 = \tilde{w}(\cdot, \tau) \tag{8.5}$$

where $\tilde{w}$ is the solution of (8.1), (8.2), (8.3) with $f \equiv 0$ which also satisfies

$$\tilde{w}(x, 0) = w_0(x) \in L^2_\rho(\Omega) \tag{8.6}$$

and we define $C = L^2(\Gamma_1 \times [0, \tau]) \to L^2_\rho(\Omega)$ by

$$Cf = \hat{w}(\cdot, \tau) \tag{8.7}$$

where $\hat{w}$ is the solution of (8.1), (8.2), (8.3) with

$$\hat{w}(x, 0) \equiv 0, \quad x \in \Omega .$$

The condition $\mathcal{R}(C) \supseteq \mathcal{R}(S)$ corresponds to controllability of (8.1), (8.2), (8.3) from an arbitrary initial state $w_0 \in L^2_\rho(\Omega)$ to the zero terminal state while $\overline{\mathcal{R}(C)} \supseteq \mathcal{R}(S)$ corresponds to approximate controllability. We easily see, just as in Section 7, that

$$S^* z_0 = z(\cdot, \tau) \tag{8.8}$$

where z satisfies (8.1), (8.2), (8.3) with $f = 0$ and

$$C^* z_0 = z\Big|_{\Gamma_1 \times [0, \tau]} \tag{8.9}$$

the right hand side of (8.9) denoting the restriction of the solution to the surface $\Gamma_1 \times [0, \tau]$ in R^{n+1}.

Via duality, the condition for approximate controllability becomes the distinguishability condition

$$C^* z_0 = z\Big|_{\Gamma_1 \times [0, \tau]} = 0 \Rightarrow z_0 = 0 .$$

This has been studied by Lions in [57] with the aid of a uniqueness theorem of Mizohata [69] which plays the same role here as the Holmgren-John uniqueness result of Section 5. The result is

Theorem 8.1. The control system (8.1), (8.2), (8.3) is approximately controllable in time τ if τ is any positive number and Γ_1 is an arbitrary non-empty relatively open subset of $\partial\Omega$.

We remark that, since the range of S is dense in this case there is no ambiguity as to the meaning of approximate controllability. (See the remarks in the paragraph preceding (2.16) in Section 2.)

To proceed further than this it is necessary that we carry out a more detailed analysis. It is known ([2], [10]) that the operator

$$Lw = -\frac{1}{\rho} \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(\alpha_j^i(x) \frac{\partial w}{\partial x^j} \right)$$

with homogeneous boundary conditions of the form (8.2), (8.3) (with $f = 0$) is an unbounded self-adjoint operator in $L_\rho^2(\Omega)$ having eigenvalues

$$0 < \lambda_1 < \lambda_2 \dots < \lambda_k < \lambda_{k+1} < \dots$$

with $\lim_{k \to \infty} \lambda_k = \infty$. (Actually there is an eigenvalue $\lambda_0 = 0$ with Γ_0 is empty, an important case, but to avoid confusion we will exclude this case from our general discussion here). Each λ_k has finite multiplicity m_k and corresponding eigenfunctions

$$\phi_{1,1}, \dots, \phi_{1,m_1}, \phi_{2,1}, \dots, \phi_{2,m_2}, \dots, \phi_{k,1}, \dots, \phi_{k,m_k}, \dots$$

can be selected so that they form an orthonormal basis for $L^2_\rho(\Omega)$. Each element $w \in L^2_\rho(\Omega)$ has an expansion

$$w = \sum_{k=1}^{\infty} \sum_{\ell=1}^{m_k} w_{k,\ell}\, \phi_{k,\ell}$$

with

$$w_{k,\ell} = \int_\Omega \phi_{k,\ell}(x)\, w(x)\, \rho(x) dx, \quad \sum_{k=1}^{\infty} \sum_{\ell=1}^{m_k} |w_{k,\ell}|^2 < \infty .$$

If we let $w(x,t)$ be a solution of (8.1), (8.2), (8.3) with initial state (8.6) having the expansion

$$w_0 = \sum_{k=1}^{\infty} \sum_{\ell=1}^{m_k} w_{0,k,\ell}\, \phi_{k,\ell}$$

and likewise expand

$$w(\cdot, t) = \sum_{k=1}^{\infty} \sum_{\ell=1}^{m_k} w_{k,\ell}(t)\, \phi_{k,\ell} \quad ,$$

then by choosing $z(x,t) = \phi_{k,\ell}(x)\, e^{-\lambda_k t}$ (a solution of (8.1), (8.2), (8.3) with $f = 0$), the identity (8.4) gives

$$w_{k,\ell}(\tau) = e^{-\lambda_k \tau}\, w_{0,k,\ell}$$

$$+ \int_0^\tau \int_{\Gamma_1} \phi_{k,\ell}(x)\, e^{-\lambda_k(\tau - t)}\, f(x,t) ds\, dt \quad ,$$

$$k = 1, 2, \ldots, \quad \ell = 1, 2, \ldots, m_k \quad . \tag{8.10}$$

If the initial and terminal states w_0 and $w(\cdot, \tau)$ have been determined,

(8.10) constitutes a moment problem for the unknown control function $f \in L^2(\Gamma_1 \times [0, \tau])$ of much the same character as (7.36) in the preceding section. Being able to solve (8.10) again depends on finding functions $q_{k,\ell}(x, t)$ biorthogonal with respect to the functions $p_{k,\ell}(x, t) = \phi_{k,\ell}(x) e^{-\lambda_k t}$. Once found, (8.10) has the formal solution

$$f(x, t) = \sum_{k=1}^{\infty} \sum_{\ell=1}^{m_k} (w_{k,\ell}(\tau) - e^{-\lambda_k \tau} w_{0,k,\ell}(x, \tau - t)) q_{k,\ell}(x, \tau - t) \quad (8.11)$$

and one can proceed, via estimates on $\|q_{k,\ell}\|$, to determine when this series converges in $L^2(\Gamma_1 \times [0, \tau])$.

For special pairs Ω, Γ_1 and $\rho(x) \equiv 1$, $A(x) \equiv I$, this analysis has been carried out directly. In [25] Fattorini studies the case where Ω is a multi-dimensional rectangle ("parallelopipedon") and Γ_1 is one face. That work is carried out in the controllability context. In [68] Mizel and Seidman obtain equivalent results from the standpoint of observability. In [28] Fattorini and the present author have considered the case wherein Ω is the unit ball in R^n and $\Gamma_1 = \partial\Omega$, the unit sphere. In both cases it was possible, by careful study of elementary or special functions to obtain explicit bounds on the biorthogonal functions $q_{k,\ell}$. The results of Theorem 7.3 remain valid in these cases with L taken as the Laplace operator in Ω with appropriate boundary conditions. In [102] Seidman obtains results for the case where Ω is a region in R^2 lying between two circles centered at the origin and between two rays out of the origin. We shall have more to say about this last result.

For a general configuration Ω, Γ_1 (or "setting", as Seidman calls it, [102]) the only controllability results that we have (other than the approximate controllability results of Theorem 8.1) come to us indirectly, either via the heat equation in a larger "regular" region ([100] or via the wave equation results of Section 5 followed by the Fourier transform.

In [100] Seidman has developed an "imbedding" technique applicable to the system

$$\frac{\partial w}{\partial t} - \sum_{k=1}^{n} \frac{\partial^2 w}{(\partial x^k)^2} = 0, \quad x \in \Omega, \quad t \in [0, \tau] \ , \tag{8.12}$$

$$\alpha w(x,t) + \beta \frac{\partial w}{\partial \nu}(x,t) = f(x,t), \quad x \in \Gamma, \quad t \in [0, \tau] \ . \tag{8.13}$$

Here Ω is a bounded, open, connected region in R^n with smooth boundary Γ and unit outward norm $\nu = \nu(x)$ defined for $x \in \Gamma$. This being the case, we proceed with Seidman [100] to imbed Ω in a larger region Ω_e for which the control problem for (8.12), (8.13) has already been solved. Then, given the initial state w_0, we extend w_0 to a function $w_0^e \in L^2(R)$ (e.g., by setting $w_0^e(x) = w_0(x)$, $x \in \Omega$, and $= 0$, $x \in R - \Omega$). Let $R(\Omega_e, \tau)$ denote the set of states known to be reachable for the system (8.12), (8.13) (but now in Ω_e rather than Ω). Given $w_\tau^e \in R(\Omega_e, \tau)$ we select a control, call it f_e, which steers (8.12), (8.13) (in Ω_e) from w_0^e to w_τ^e. Let the resulting solution of (8.12) in $\Omega_e \times [0, \tau]$ be called $w^e(x,t)$. Since $\partial\Omega \subseteq \Omega_e \cup \partial\Omega_e$, setting

$$f(x,t) = \frac{\partial w^e}{\partial \nu}(x,t), \quad x \in \partial\Omega, \quad t \in [0, \tau] \tag{8.14}$$

we can use standard uniqueness and regularity results to see that the control f thus derived steers (8.12), (8.13) (in Ω now) from w_0 to w^e_τ restricted to Ω. This result can be expressed by

$$R(\Omega, \tau) \supseteq R(\Omega_e, \tau)\Big|_{\Omega} .$$

Commonly one takes Ω_e to be a ball in R^n enabling one to use the controllability result in [28] or, equally well, the observability result in [68]. For such an Ω_e essentially the result of Theorem 7.3 remains valid, as demonstrated in [100].

These imbedding methods suffer from the limitation that the control must be applied along the whole boundary unless we have $\Gamma = \Gamma_1 \cap \Gamma_0$ with Γ_0 so configured that repeated reflections produce a larger region whose boundary consists entirely of Γ_1 and its reflected images. This means, effectively that Γ_0 consists of two line segments meeting at a point with the angle between them equal to $2\pi/m$ for some appropriate integer m.

We begin the rest of the work of this section with an appreciation of what Theorem 5.3 means in terms of the harmonic analysis of the functions

$$\hat{p}_{k,\ell}(x,t) = \phi_{|k|,\ell}(x)\, e^{i\omega_k t}, \quad x \in \Gamma_1, \quad t \in [0, T] ,$$

$$k = \pm 1, \pm 2, \pm 3, \ldots, \qquad \ell = 1, 2, \ldots, m_{|k|} .$$

which are restrictions to $\Gamma \times [0,T]$ of "eigenfunction" solutions of the system (5.25), (5.26), (5.27) with $u \equiv 0$. We are supposing that $T > T_1$ (cf. Theorem 5.3) and we are adopting the notation $\omega_k = \sqrt{\lambda_k}$, $k = 1, 2, 3, \ldots$, $\omega_k = -\sqrt{\lambda_{-k}}$, $k = -1, -2, -3, \ldots$. The numbers $-\lambda_k$ are the eigenvalues of the Laplacian in Ω with homogeneous boundary conditions of the form (5.26), (5.27). With $\phi_{k,\ell}$, $k = 1, 2, 3, \ldots$, $\ell = 1, 2, \ldots, m_k$ being the corresponding orthonormal eigenfunctions that states

$$\Phi_{k,\ell} = \frac{1}{2}\left(\frac{1}{i\omega_k}\phi_{|k|,\ell},\ \phi_{|k|,\ell}\right), \quad k = \pm 1, \pm 2, \pm 3, \ldots, \quad \ell = 1, 2, \ldots, m_{|k|}, \tag{8.15}$$

form a complete orthonormal set in the space $\mathscr{K}_E(\Omega) =$ (cf. (5.9)) Z equipped with the energy inner product

$$\langle ((w,v); (\hat{w}, \hat{v}))\rangle_E = \int_\Omega \left(v\,\bar{\hat{v}} + \sum_{j=1}^n \frac{\partial w}{\partial x^j}\frac{\partial \bar{\hat{w}}}{\partial x^j}\right)dx .$$

Using (5.17) (modified for complex solutions) with

$$(w(\cdot,0),\ v(\cdot,0)) = 0, \quad (w(\cdot,T),\ v(\cdot,T)) = \Phi_{\hat{k},\hat{\ell}} \tag{8.16}$$

$$(y(\cdot,T),\ z(\cdot,T)) = \Phi_{k,\ell}, \quad (y(\cdot,0),\ z(\cdot,0)) = e^{-i\omega_k T}\,\Phi_{k,\ell}$$

and letting $\hat{q}_{k,\ell} = \hat{q}_{k,\ell}(x,t)$, $x \in \Gamma_1$, $t \in [0,T]$ be a control (cf. (5.27)) in $L^2(\Gamma_1 \times [0,T]$ which solves the control problem (5.25), (5.26), (5.27) (8.16) we find that

$$\langle \Phi_{\hat{k},\hat{\ell}}, \Phi_{k,\ell} \rangle_E = \delta_{k,\ell}^{\hat{k},\hat{\ell}} = \int_0^T \int_{\Gamma_1} \phi_{|k|,\ell}(x) \, e^{-i\omega_k (T-t)} \hat{q}_{\hat{k},\hat{\ell}}(x,t) \, ds \, dt$$

$$= (q_{\hat{k},\hat{\ell}}, p_{k,\ell})_{L^2(\Gamma_1 \times [0,T])} \, , \tag{8.17}$$

so that the $q_{k,\ell}$ form a biorthogonal set relative to the $p_{k,\ell}$ in $L^2(\Gamma_1 \times [0,T])$. This result can be compared, qualitatively, with the results on nonharmonic Fourier series in [73], [56], [99], [46] (see [87] for summary) and is obtained by what is seen to be the method described very briefly following (4.25) in Section 4. We remark that one can show by methods analogous to those referred to in connection with Theorem 5.2 that there is a $T_0 \leq T_1$ such that the control problem is not solvable, and hence biorthogonal functions $q_{k,\ell}$ do not exist in $L^2(\Gamma_1 \times [0,T])$, if $T < T_0$ — again similar to the above noted results in nonharmonic Fourier series. The positive result for $T > T_1$ is only qualitatively similar to the classical results in nonharmonic Fourier series because we cannot establish (even though we do conjecture it to be true) that the $p_{k,\ell}$ and $q_{k,\ell}$ are Riesz bases for the subspaces of $L^2(\Gamma_1 \times [0,T])$ which they span. (These subspaces can, w.l.o.g., be assumed identical by taking the $q_{k,\ell}$ to be controls of least $L^2(\Gamma_1 \times [0,T])$ norm. See [57] for existence theorems in this connection.) What one can establish (see [92], [93] is that for some positive integer r and some $M > 0$

$$\|\hat{q}_{k,\ell}\|_{C(\Gamma_1 \times [0,T])} \leq M \omega_{|k|}^r \, .$$

One may also arrange (as in [92] that

$$\hat{q}_{k,\ell}(x, t) = \overline{\hat{q}_{-k,\ell}(x, t)} \quad ,$$

as one would expect from (8.15) and the definition of ω_k, $k = -1, -2, -3, \ldots$.
We will assume this has been done.

Let functions $\hat{G}_{k,\ell}(x, z)$ be defined for $x \in \Gamma_1$ and z in the complex plane via the Fourier transform

$$\hat{G}_{k,\ell}(x, z) = \int_0^T e^{-izt}\,\hat{q}_{k,\ell}(x, T-t)dt \quad .$$

The $\hat{G}_{k,\ell}(x, z)$ are of class C in x and entire in z. Moreover, one may show very easily that the $\hat{G}_{k,\ell}(x, z)$ are of uniform exponential type:

$$|\hat{G}_{k,\ell}(x, z)| \leq M_0\, e^{|z|T}\,\omega^r_{|k|} \tag{8.18}$$

for some $M_0 > 0$.

The biorthogonality relation (8.17) now becomes (after a trivial change of variable)

$$\int_{\Gamma_1} \phi_{|k|,\ell}(x)\,\hat{G}_{\hat{k},\hat{\ell}}(x, \omega_k)ds = \delta^{\hat{k},\hat{\ell}}_{k,\ell} \quad . \tag{8.19}$$

$$k, \hat{k} = \pm 1, \pm 2, \pm 3, \ldots, \qquad \ell, \hat{\ell} = 1, 2, \ldots, m_{|k|}, m_{|\hat{k}|} \quad .$$

Now, using only the $\hat{G}_{k,\ell}$ for $k = 1, 2, \ldots$ (no negative values of k) and $\ell = 1, 2, \ldots, m_k$ we find, by setting

$$\tilde{G}_{k,\ell}(x, z) = \hat{G}_{k,\ell}(x, z) + \hat{G}_{k,\ell}(x, -z)$$

that we can obtain a function, again with growth like (8.18), but now an even function of z. Moreover, (8.19) gives

$$\int_{\Gamma_1} \phi_{k,\ell}(x) \quad \tilde{G}_{\hat{k},\hat{\ell}}(x, \pm \omega_k)\, dx = \delta_{k,\ell}^{\hat{k},\hat{\ell}} \tag{8.20}$$

$$k, \hat{k} = 1, 2, 3, \ldots, \qquad \ell, \hat{\ell} = 1, 2, \ldots, m_k, \; m_{\hat{k}} .$$

Since $\tilde{G}$ is even we may define

$$G_{k,\ell}(x, -iz^2) = \tilde{G}_{k,\ell}(x, z), \qquad k = 1, 2, 3, \ldots, \quad \ell = 1, 2, \ldots, m_k$$

and obtain again a function of class $\mathcal{C}$ with respect to x and entire in z, now with

$$|G_{k,\ell}(x, z)| \leq 2M_0 \, e^{|z|^{\frac{1}{2}\tau}} \, \omega_k^r . \tag{8.21}$$

Then (8.20) becomes

$$\delta_{k,\ell}^{\hat{k},\hat{\ell}} = \int_{\Gamma_1} \phi_{k,\ell}(x) \quad G_{\hat{k},\hat{\ell}}(x, -i\lambda_k)\, ds . \tag{8.22}$$

The next step would be to infer that $G_{\hat{k},\hat{\ell}}(x, z)$ is the Fourier transform of some appropriate function. But here we are stopped momentarily because (8.21) indicates $G_{\hat{k},\hat{\ell}}(x, z)$ to be the Fourier transform of a distribution with support at $t = 0$ – not a bona fide function at all. This situation is remedied with use of a theorem [83] of Redheffer. $G_{\hat{k},\hat{\ell}}(x, z)$ is multiplied by an entire function $E_\tau(z)$ designed so that $E_\tau(z)\, G_{\hat{k},\hat{\ell}}(x, z)$ is the Fourier transform of a function $\tilde{q}_{\hat{k},\hat{\ell}}(x, t)$ in $L^2(\Gamma_1 \otimes [0, \tau])$ – this can be done for any $\tau > 0$. Then (8.22) reads

$$E_\tau(-i\lambda_k)\,\delta_{k,\ell}^{\hat{k},\hat{\ell}} = \int_{\Gamma_1} \phi_{k,\ell}(x)\quad E_\tau(-i\lambda_k)\, G_{\hat{k},\hat{\ell}}(x, -i\lambda_k)\, ds$$

$$= \int_0^\tau \int_{\Gamma_1} \phi_{k,\ell}(x)\quad e^{-\lambda_k t}\quad q_{\hat{k},\hat{\ell}}(x,t)\, ds\, dt$$

$$= (p_{k,\ell},\ \tilde{q}_{\hat{k},\hat{\ell}})_{L^2(\Gamma_1 \times [0,\tau])}\ ,$$

where the definition of $p_{k,\ell}$ should be clear. Hence

$$q_{k,\ell} = \frac{\tilde{q}_{k,\ell}}{E_\tau(-i\lambda_k)}$$

satisfies the biorthogonality relation

$$(p_{k,\ell},\ q_{\hat{k},\hat{\ell}})_{L^2(\Gamma_1 \times [0,\tau]} = \delta_{k,\ell}^{\hat{k},\hat{\ell}}\ .$$

The moment problem (8.10) then has the formal solution (8.11). By careful estimation of the numbers $E_\tau(-i\lambda_k)$ one can establish

$$\|q_{k,\ell}\|_{L^2(\Gamma_1 \times [0,\tau]} \le \tilde{K}_0\ \exp(\tilde{K}_1\ \omega_k)\ , \tag{8.23}$$

$$k = 1, 2, 3, \ldots, \qquad \ell = 1, 2, \ldots, m_k$$

just as in (7.33). Theorem 7.3 is then valid (with the obvious modifications) for the control system (8.1), (8.2), (8.3) with $\rho(x) \equiv 1$, $A(x) \equiv I$ and the pair (Ω, Γ_1) star-complemented. This last point should be re-emphasized – the controllability result just obtained applies only to star-complemented pairs (Ω, Γ_1), since that assumption is vital in our proof of Theorem 5.3 which, in turn, provides the formulation for the

result just obtained. Now the example in Section 5 of the rectangle with control applied on only one side shows that controllability of finite energy states with controls $u \in L^2(\Gamma_1 \times [0, \tau])$ cannot be generally expected for the wave equation without the condition on star-complementation (or (see [72], [106]) some other condition which guarantees that the "inactive" portion, Γ_0, of $\partial\Omega$ does not "trap waves". Because wave fronts travel along straight line "rays" ([11]) in the case of the wave equation this is not surprizing. But one can make no convincing case that such geometric conditions are in any way necessary for parabolic equations modelling heat or diffusion processes. A recent example due to Seidman [102] shows, in fact, that parabolic processes may exhibit a significant degree of controllability even when the pair Ω, Γ_1 is not star-complemented.

To explain this result of Seidman we use a rather less general system than he did in [102]. We consider

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} = 0, \quad 0, \quad 0 \le t \le \tau, \quad (x, y) \in \Omega \tag{8.24}$$

$$w(x, y, t) = 0, \ (x, y) \in \Gamma_0, \ w(x, y, t) = f(x, y), \quad (x, y) \in \Gamma_1, \ 0 \le t \le \tau \ , \tag{8.25}$$

where Ω is the annular region

$$\Omega = \{(x, y) = (r \cos \theta, \ r \sin \theta) \,|\, r_0 < r < r_1\} \ .$$

If we take

$$\Gamma_1 = \{(x, y) \mid r = r_1\}, \quad \Gamma_0 = \{(x, y) \mid r = r_0\} , \tag{8.26}$$

then the pair Ω, Γ_1 is star-complemented $(\Omega^* = \{(x, y) \mid r < r_0\}$. On the other hand, if we take

$$\Gamma_1 = \{(x, y) \mid r = r_0\}, \quad \Gamma_0 = \{(x, y) \mid r = r_1\} \tag{8.27}$$

then Ω, Γ_1 is not star-complemented.

Following the duality approach used repeatedly in earlier parts of this paper controllability of (8.24), (8.25) is related to the observability of

$$\frac{\partial z}{\partial t} - \frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0, \quad 0 \leq t \leq \tau, \quad (x, y) \in \Omega , \tag{8.28}$$

$$z(x, y, t) = 0, \quad (x, y) \in \Gamma_0, \quad z(x, y, t) = 0, \quad (x, y) \in \Gamma_1 \tag{8.29}$$

$$\omega(x, y, t) \equiv \frac{\partial z}{\partial \nu}(x, y, t) \left(\equiv \frac{\partial z}{\partial r}(r, \theta)\right), \quad (x, y) \in \Gamma_1 \quad (r = r_1 \text{ (or } r_0)) . \tag{8.30}$$

We will consider first the case (8.26) where Ω, Γ_1 is star-complemented. Here our earlier controllability results apply to give controllability of (8.24), (8.25) for arbitrary $\tau > 0$. To express this in terms of bounds on biorthogonal functions we note that the eigenfunctions of the Laplacian in Ω with homogeneous Dirichlet boundary data have the form

$$\phi_{k,\ell}(r, \theta) = e^{ik\theta} z_{k,\ell}(r), \quad k = 0, 1, 2, 3, \ldots, \quad \ell = 1, 2, 3, \ldots$$

where the $z_{k,\ell}$ are the solutions of the following eigenvalue problem for Bessel's equation.

$$\frac{d^2 z_{k,\ell}}{dr^2} + \frac{1}{r}\frac{dz_{k,\ell}}{dr} + (\lambda_{k,\ell} - \frac{k^2}{r^2}) z_{k,\ell} = 0, \quad r_0 < r < r_1 \tag{8.31}$$

$$z_{k,\ell}(r_0) = z_{k,\ell}(r_1) = 0 \ . \tag{8.32}$$

Assuming (8.31), (8.32) has been solved to obtain the $\lambda_{k,\ell}$ and $\phi_{k,\ell}$, a solution of (8.28), (8.29) has the form

$$z(x,y,t) = z(r,\theta,t) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{\infty} \zeta_{k,\ell}\, e^{-\lambda_{k,\ell} t}\, \phi_{k,\ell}(r,\theta)$$

$$= \sum_{k=0}^{\infty} \sum_{\ell=1}^{\infty} \zeta_{k,\ell}\, e^{-\lambda_{k,\ell} t}\, e^{ik\theta} z_{k,\ell}(r) \ .$$

The observation (8.30) is (continuing to assume (8.26))

$$\omega(x,y,t) = \omega(\theta,t) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{\infty} \zeta_{k,\ell}\, e^{-\lambda_{k,\ell} t}\, e^{ik\theta} \frac{\partial z_{k,\ell}}{\partial r}(r_1) \ .$$

$$\equiv \sum_{k=0}^{\infty} \sum_{\ell=1}^{\infty} \zeta_{k,\ell}\, p_{k,\ell}(\theta,t) \ . \tag{8.33}$$

The foregoing theory of controllability for star-complemented configurations shows that the set of functions $\{p_{k,\ell}\}$ possesses a biorthogonal set in $L^2(\Gamma_1 \times [0,\tau])$, namely the $\{q_{k,\ell}\}$ for which we have bounds (cf. (8.23))

$$\| q_{k,\ell} \|_{L^2(\Gamma_1 \times [0,\tau])} \le \tilde{K}_0 \exp(\tilde{K}_1 \omega_{k,\ell}) \ . \tag{8.34}$$

(Note that the indices k, ℓ are used differently in this example than they are in the analysis leading up to (8.23).) The result of Theorem 7.3

depends on the term $\omega_{k,\ell} = \sqrt{\lambda_{k,\ell}}$ appearing in the argument of the exponential in (8.23), (8.34).

If we now pass to the case (8.27) the only change is that (8.33) becomes

$$\begin{aligned}\tilde{\omega}(x, y, t) = \tilde{\omega}(0, t) &= \sum_{k=0}^{\infty} \sum_{\ell=1}^{\infty} \zeta_{k,\ell}\, e^{-\lambda_{k,\ell} t} e^{ik\theta} \frac{\partial z_{k,\ell}}{\partial r}(r_0) \\ &= \sum_{k=0}^{\infty} \sum_{\ell=1}^{\infty} \zeta_{k,\ell} \left(\frac{\partial z_{k,\ell}}{\partial r}(r_0) \Big/ \frac{\partial z_{k,\ell}}{\partial r}(r_1)\right) e^{-\lambda_{k,e} t} e^{ik\theta} \frac{\partial z_{k,\ell}}{\partial r}(r_1) \\ &= \sum_{k=0}^{\infty} \sum_{\ell=1}^{\infty} \zeta_{k,\ell} \left(\frac{\partial z_{k,\ell}}{\partial r}(r_0) \Big/ \frac{\partial z_{k,\ell}}{\partial r}(r_1)\right) p_{k,\ell}(\theta, t) \\ &= \sum_{k=0}^{\infty} \sum_{\ell=1}^{\infty} \zeta_{k,\ell}\, \tilde{p}_{k,\ell}(\theta, t) .\end{aligned}$$

The set of functions $\{\tilde{p}_{k,\ell}\}$ *clearly* has in $L^2(\Gamma_1 \times [0, \tau])$ the biorthogonal set $\{\tilde{q}_{k,\ell}\}$:

$$\tilde{q}_{k,\ell} = \frac{r_1}{r_0} \left(\frac{\partial z_{k,\ell}}{\partial r}(r_1) \Big/ \frac{\partial z_{k,\ell}}{\partial r}(r_0)\right) a_{k,\ell} . \tag{8.35}$$

To obtain an estimate of the form (8.34) for the $\tilde{q}_{k,\ell}$ we need bounds on

$$\frac{\partial z_{k,\ell}}{\partial r}(r_1) \Big/ \frac{\partial z_{k,\ell}}{\partial r}(r_0) .$$

Such an estimate has been obtained in [102]. Multiplying (ω. 31) by r^2 and then putting $s = \log r$, (8.31) becomes

$$\frac{d^2 z_{k,\ell}}{ds^2} + (e^{2s} \lambda_{k,\ell} - k^2) z_{k,\ell} = 0 \tag{8.36}$$

$$\log r_0 < s < \log r_1 .$$

Positivity of the operator $-\frac{d^2 z}{ds^2} + k^2 z$ implies we must have, for all k, ℓ,

$$r_1^2 \lambda_{k,\ell} \geq k^2 . \tag{8.37}$$

We may assume therefore that (8.36) takes the form

$$\frac{d^2 z_{k,\ell}}{ds^2} + (f_{k,\ell}(s) \lambda_{k,\ell} + 1) z_{k,\ell}$$

$$\log r_0 < s < \log r_1$$

where $f_{k,\ell}(s)$ is bounded, uniformly in k, ℓ and s, and monotone (increasing) as a function of s for each k, ℓ. Using the differential equation in this form, a rather nice argument in [102] establishes that

$$\left| \frac{\partial z_{k,\ell}}{\partial r}(r_1) \Big/ \frac{\partial z_{k,\ell}}{\partial r}(r_0) \right| \leq \hat{K}_0 \exp(\hat{K}_1 \omega_{k,\ell})$$

with $\hat{K}_0$, $\hat{K}_1$ depending on the bound satisfied by $|f(s)|$, r_1 and r_0. From this and (8.34) it is clear that the $\tilde{q}_{k,\ell}$ do satisfy an inequality again of the form (8.34) and one obtains the result of Theorem 7.3 for the case (8.27) via the route (8.10), (8.11) again – this time without any condition of star-complementation.

No complete characterization of settings for which the results of Theorem 7.3 (restated for (8.1), (8.2), (8.3)) obtain is available at this writing.

9. RELATED TOPICS

In this concluding section we will consider three topics which are not, in the strict sense, included in the area of inquiry announced in the title of this article. They are: controllability of nonlinear systems; the linear-quadratic regulator theory for linear distributed parameter systems; and the time optimal control problem for linear distributed parameter systems. They are legitimate subsidiary topics for this sort of review paper because developments in these areas are essentially dependent upon the existence of an adequate controllability and stabilizability theory for related linear systems. We can give only the briefest idea of the nature of existing and developing results because those results are, for the most part, incomplete and fragmented and because of space limitations. There are, of course, other topics, e.g., the representation theories of Baras and Brockett ([3], [4]), Helton ([35], [36]) and others which are also related to controllability. We do not attempt to treat these in the present article.

9a. Controllability of Nonlinear Systems

We have about fifteen years of history for controllability of a nonlinear finite dimensional system

$$\dot{x} = f(x, u), \quad x \in E^n, \quad u \in E^m . \tag{9.1}$$

Supposing that $f = E^{m+n} \rightarrow E^n$ is of class C^1 with $f(0,0) = 0$, one seeks to investigate controllability of (9.1) for (x, u) near $(0,0)$ under the basic assumption of controllability for the linearized system

$$\dot{x} = Ax + Bu, \quad A = \frac{\partial f}{\partial x}(0,0), \quad B = \frac{\partial f}{\partial u}(0,0) \quad . \tag{9.2}$$

The earliest definitive result appears to be the paper [65] of Lee and Markus appearing in 1961. A somewhat "streamlined" version appears in their book [55]. We will use their approach, based on the implicit function theorem, but will couch it in the setting (developed in [47], [61]) which we have already introduced in Section 2.

Consider the problem of "reachability". We let $x(0) = 0$, for $T > 0$, and consider those states x_1 which can be reached at time T with $u \in L^2([0,T]; E^m)$. We have

$$x_1 = x(T) = \int_0^T e^{A(T-t)} B\, u(t)dt \quad . \tag{9.3}$$

We consider a control u depending on an n-dimensional parameter vector ξ :

$$u(\xi, t) = B^* e^{A^*(T-t)} \xi \equiv U(t)\xi \quad . \tag{9.4}$$

Substituting (9.4) in (9.3) and solving for ξ we have

$$\xi = Z(T)^{-1} x_1 \quad ,$$

where $Z(T)$ is the matrix in (2.10), whose inverse exists just in case (9.2) is controllable. This establishes a nonsingular linear transformation

$$x_1 = x_1(\xi) = Z(T)\xi$$

between the state x_1 and the parameter ξ determining the control $u = u(\xi, t)$.

Writing (9.1) in the form

$$\dot{x} = Ax + Bu + g(x, u) \quad ,$$

with $g(x, u) = f(x, u) - Ax - Bu$, we have

$$x_1 = \int_0^T e^{A(T-t)}(Bu(t) + g(x(t), u(t)))dt \quad .$$

Let us again write $u = u(\xi, t)$ in the form (9.4). We then have

$$\begin{aligned} x_1 &= \int_0^T e^{A(T-t)}(B\,U(t)\xi + g(x(\xi, t), U(t)\xi)dt \\ &\equiv F(\xi) \quad , \end{aligned} \tag{9.5}$$

with $x(\xi, t)$ the solution of

$$\frac{\partial x(\xi, t)}{\partial t} = f(x(\xi, t), U(t)\xi), \quad x(\xi, t) = 0 \quad .$$

Following [55] and [61] we compute

$$\frac{\partial F}{\partial \xi} = \int_0^T e^{A(T-t)} B\,U(t)dt$$

$$+ \int_0^T e^{A(T-t)}[\frac{\partial g}{\partial x}(x(\xi, t), U(t)\xi) \frac{\partial x}{\partial \xi}(\xi, t) + \frac{\partial g}{\partial u}(x(\xi, t), U(t)\xi) U(t)]dt$$

When $\xi = 0$ we clearly have $x(\xi, t) \equiv 0$, $U(t)\xi \equiv 0$ and, since $\frac{\partial g}{\partial x}(0, 0) = 0$, $\frac{\partial g}{\partial u}(0, 0) = 0$, we find that

$$\begin{aligned} \frac{\partial F}{\partial \xi}(0) &= \int_0^T e^{A(T-t)} B\,U(t)dt = (\text{cf. } (9.4)) \\ &= \int_0^T e^{A(T-t)} BB^* e^{A^*(T-t)} dt = Z(T) \quad , \end{aligned}$$

i.e., the Jacobian of the map (9.5) at $\xi = 0$ is the map $Z(T)$ associated with the linearized system (9.2). The implicit function theorem shows immediately that (9.5) is solvable, yielding

$$\xi = F^{-1}(x_1)$$

for x_1 in some neighborhood of 0, showing that (9.2) remains controllable in the sense that it can be steered from the origin to any sufficiently small state x_1 during $[0, T]$.

Most existing results on controllability of nonlinear distributed systems follow this same pattern. A control to state (or control parameter to state) map is constructed for the nonlinear system and its Jacobian is shown to be the comparable map for the linearized system.

One exception to this rule is the paper [9] of M. Cirina. In it he considers a quasilinear generalization of the hyperbolic system studied here in Section 3, viz.

$$\frac{\partial w}{\partial t} = A(x, t, w) \frac{\partial w}{\partial x} + f(x, t, w) \quad . \tag{9.6}$$

In his work controls are applied at $x = 0$ and $x = 1$ via (cf. (3.11), (3.12), (3.13))

$$w^-(0, t) = u(t), \quad w^+(1, t) = v(t) \tag{9.7}$$

or else just at one end:

$$w^-(0, t) = 0, \quad w^+(1, t) = v(t) \; . \tag{9.8}$$

In either case Cirina considers the "null-controllability" problem

$$w(x, 0) = \phi(x), \quad w(x, T) = 0$$

since (9.6), (9.7), (9.8) are not time reversible. However one may modify the method to allow $w(x, 0)$, $w(x, T)$ to both be given arbitrarily (within an appropriate class) in the case of time reversible systems. Theorem 3.2, with certain modifications so that the Goursat problem is replaced by a standard initial-boundary value problem for the equation obtained from (9.6) by reversing the roles of x and t:

$$\frac{\partial w}{\partial x} = A(x, t, w)^{-1}\left[\frac{\partial w}{\partial t} - f(x, t, w)\right] .$$

Exact controllability results comparable to those of Theorem 3.2 are obtained constructively, for initial states $\phi \in C^1([0, 1]; E^m)$ with sufficiently small norm.

Cirina's result is not based on the implicit function theorem - and probably could not be. A fair description of his method would seem to be this. Theorem 3.2 provides a method whereby one can construct a map from $w(\cdot, 0)$ to a control u (or control pair u, v) steering $w(\cdot, 0)$ to zero at time T. (Theorem 3.2 can be modified to treat controls applied at $x = 0$ and $x = 1$; see [86].) Cirina's contribution consists in his establishing that this mapping procedure can be carried over to the quasilinear systems (9.6). The method is more direct than the implicit function method which first constructs the nonlinear map F (cf. (9.5)) and then infers the existence of a nonlinear F^{-1} based on properties of the Jacobian of F. The method is heavily dependent

upon the fact that the roles played by x, t in hyperbolic systems in two variables are interchangeable and thus appears not to be extendable to other types of systems.

The first use of implicit function type methods to establish controllability of nonlinear distributed parameter systems appears to be due to H. O. Fattorini [24]. He considers a nonlinear wave equation

$$\rho(x) \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(a(x) \frac{\partial w}{\partial x}\right) - F(x, w) + g(x)\, u(t) \quad ,$$

$$0 \le x \le 1, \quad 0 \le t \le T, \quad \rho(x) \ge \rho_0 > 0, \quad a(x) \ge a_0 > 0 \quad .$$

The basic assumption on F is that it is of class C^1 and odd as a function of w with $\frac{\partial F}{\partial w}(x, 0) \ge 0$. The boundary conditions initially take the form

$$a_0\, w(0, t) + b_0 \frac{\partial w}{\partial x}(0, t) = 0, \qquad (a_0)^2 + (b_0)^2 \ne 0 \quad ,$$

$$a_1\, w(1, t) + b_1 \frac{\partial w}{\partial x}(1, t) = 0 \qquad (a_1)^2 + (b_1)^2 \ne 0 \quad ,$$

but other restrictions are imposed in the course of the work. The problem treated is that of reachability:

$$w(\cdot, 0) = \frac{\partial w}{\partial t}(\cdot, 0) = 0, \quad w(\cdot, T) = w_1, \quad \frac{\partial w}{\partial t}(\cdot, T) = v_1 \quad .$$

It is studied with reference to the same problem as posed for the linearized equation

$$\rho(x) \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(a(x) \frac{\partial w}{\partial x}\right) - \frac{\partial F}{\partial w}(x, 0) w + g(x)\, u(t) \quad . \tag{9.9}$$

This problem can be resolved by the method presented at the beginning of Section 4, involving a moment problem in the theory of nonharmonic Fourier series.

The operator

$$L = \frac{1}{\rho(x)}\left[-\frac{\partial}{\partial x}\left(a(x)\frac{\partial w}{\partial x}\right) + \frac{\partial F}{\partial w}(x,\cdot)w\right]$$

with the indicated boundary conditions is unbounded, positive, self-adjoint in $L^2_\rho[0,1]$, the eigenfunctions ϕ_k forming an orthonormal basis for $L^2_\rho[0,1]$. (Recall the inner product in $L^2_\rho[0,1]$ is $(w,\tilde{w})_\rho = \int_0^1 w(x)\,\tilde{w}(x)\,\rho(x)dx$.) Expanding g:

$$g = \sum_{k=1}^{\infty} g_k\,\phi_k$$

one requires

$$g_k \neq 0, \quad k = 1, 2, 3, \ldots$$

$$\liminf_{k\to\infty}\, kg_k = \tilde{g} > 0 \quad .$$

$$\limsup_{k\to\infty}\, kg_k = \hat{g} > 0 \quad .$$

Under these assumptions it is shown in [24] that all states $(w_1, v_1) \in H \times K$ are reachable, where

$$H = \{w \in \mathrm{Dom}(L)\}$$

$$K = \{w \in \mathrm{Dom}(L^{\frac{1}{2}})\}$$

equipped with the inner products

$$(w, \tilde{w})_H = (Lw, L\tilde{w})_\rho \quad ,$$

$$(v, \tilde{v})_K = (L^{\frac{1}{2}} w, L^{\frac{1}{2}} \tilde{w})_\rho \quad ,$$

provided $T \geq T_1$, the minimal control time discussed in earlier sections. For $T = T_1$ the control to state map

$$C: \; u \in L^2[0, T] \to (w, v) \in H \times K$$

is a Hilbert space isomorphism.

The main part of the work consists in establishing that solutions of the nonlinear system (9.9) continue to lie in $H \times K$ and that the nonlinear control to state map has C, the control to state map for the linearized system, as its Jacobian. Once this has been done familiar extensions of the implicit function theorem to infinite dimensional spaces (see e.g. [13]) apply to show that the nonlinear control to state map carries a small neighborhood of 0 in $L^2[0, T]$ onto a small neighborhood of $(0, 0)$ in $H \times K$ – thus establishing local reachability for the nonlinear system.

This approach was carried further by the late W. C. Chewning [8] who applied it to a higher dimensional nonlinear hyperbolic equation with boundary value control – the linearized version of which is studied in Section 5 of this review article. The basic approach is the same as Fattorini's but there are significant differences in the assumptions which have to be made on the nonlinearities involved. Chewning's system takes the form

$$\frac{\partial^2 w}{\partial t^2} - \sum_{k=1}^{n} \frac{\partial^2 w}{(\partial x^k)^2} - f(w, \frac{\partial w}{\partial t})(x,t) = 0, \quad t \geq 0, \quad x \in \Omega$$

where Ω is a domain of the type discussed in Section 5. (Actually Ω is taken to be a higher dimensional rectangle or "parallelopipedon" in [8] but this is unnecessary.) The control takes the form

$$w(x,t) = u(x,t), \quad x \in \partial\Omega, \quad t \in [0,T]$$

for some $T > 0$. The analysis is carried out in the space $\mathcal{V}^2(\Omega)$ (cf. (5.28)) and the principal assumptions on $f: \mathcal{V}^2(\Omega) \to H^1(\Omega)$ (which may or may not take the form $f(w(x,t), \frac{\partial w}{\partial t}(x,t)))$ are (i) f is continuous and continuously differentiable in a neighborhood of $(0,0)$ in $\mathcal{V}^2(\Omega)$; (ii) $f(0,0) = 0$; (iii) $\|f(w,v) - f(\hat{w},\hat{v})\| \leq \|(w,v) - (\hat{w},\hat{v})\| \cdot C(\|(w,v)\|, \|(\hat{w},\hat{v})\|)$; and (iv) the Fréchel derivative of f is locally Lipschitz near the origin. If f has the form $f(w(x,t), \frac{\partial w}{\partial t}(x,t))$ all those conditions are satisfied if f has Lipschitz continuous partial derivatives of order ≤ 2. These assumptions permit one to again establish that the control to state map for the nonlinear system has the control to state map of the linearized system as its Jacobian. That the latter maps onto $\mathcal{V}^2(\Omega)$ is established as indicated in Section 5. The implicit function theorem again shows that the map for the nonlinear system has range which includes a neighborhood of the origin in $\mathcal{V}^2(\Omega)$.

Concerning exact controllability of nonlinear parabolic systems very little, if anything, has been reported. The implicit function approach appears to be unusable because it is very difficult to identify

a topology for the state space so that the control to state map for the linearized system is continuous and onto while, at the same time, state perturbations due to nonlinerities continue to lie in this space. This quest is made doubly difficult by the fact, noted in Sections 7, 8 that we have no adequate characterization of the set of reachable states for parabolic control processes.

In the case of approximate controllability of nonlinear parabolic equations we are more fortunate. In a recent paper, [37]. M. Henry has introduced a number of techniques which he shows to be useful in such studies and which appear to hold considerable promise for wider application. The technique is basically one of "cancellation", the control being used to steer the linearized system and, at the same time, "cancel" the state perturbation due to the nonlinearity present in the system – at least approximately. Various approximate controllability theorems are given, corresponding to a number of different control situations. We will provide the barest outline of one of them in a setting somewhat oversimplified as compared with what Henry actually employs.

Consider then the equation of Section 7, modified now to

$$\rho(x) \frac{\partial w}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} \left(a_j^i(x) \frac{\partial w}{\partial x^j}\right) + f(w(\cdot, t))(x) = 0, \quad x \in \Omega, \ t \geq 0 \ , \qquad (9.10)$$

with boundary conditions

$$\left(\frac{\partial w}{\partial x}(x,t), \ A(x)\,\nu(x)\right)_{E^n} = \begin{cases} 0 & x \in \Gamma_0 \\ u(x,t), & x \in \Gamma_1 \end{cases} \qquad (9.11)$$

and initial state

$$w(x, 0) \equiv 0 \ .$$

The assumption on $f: L^2_\rho(\Omega) \to L^2_\rho(\Omega)$ is that it should be continuous and bounded. Approximate controllability of the system with $f \equiv 0$ for controls $u \in L^2(\Gamma_1 \times [0, \tau])$, $\tau > 0$, follows from Fattorini's result which we have summarized here in Theorem 8.1. Thus, given a desired state $w_1 \in L^2_\rho(\Omega)$, and a "tolerance" $\varepsilon > 0$, we can find a control $u = u_\varepsilon$ such that

$$\| v(\cdot, \tau) - w_1 \|_{L^2_\rho(\Omega)} < \varepsilon \ , \tag{9.12}$$

v satisfying (9.10), (9.11) (with w replaced by v, f replaced by zero).

Now for $z \in L^2([0, \tau]; L^2_\rho(\Omega))$ consider the linear nonhomogeneous equation

$$\rho(x) \frac{\partial y}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} (a^i_j(x) \frac{\partial y}{\partial x^j}) + f(z(\cdot, t))(x) = 0 \tag{9.13}$$

with the same boundary conditions as above. Writing $y = \hat{y} + \tilde{y}$ where

$$\rho(x) \frac{\partial \hat{y}}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} (a^i_j(x) \frac{\partial \hat{y}}{\partial x^j}) + f(z(\cdot, t))(x) = 0$$

$$\hat{y}(\cdot, 0) = 0, \quad (\frac{\partial \hat{y}}{\partial x}(x, t), \ A(x)\, \nu(x))_{E^n} = 0, \quad x \in \Gamma, \quad t \geq 0 \ ,$$

$$\rho(x) \frac{\partial \tilde{y}}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x^i} (a^i_j(x) \frac{\partial \tilde{y}}{\partial x^i}) = 0$$

$$\tilde{y}(\cdot, 0) = 0, \quad (\frac{\partial \tilde{y}}{\partial x}(x, t), \ A(x)\, \nu(x))_{E^n} = \begin{cases} 0 & x \in \Gamma_0, \quad t \geq 0 \\ \tilde{u}(x, t), & x \in \Gamma_1, \quad t \geq 0 \end{cases}$$

we see that we can achieve

$$\|y(\cdot, \tau) - w_1\|_{L^2_\rho(\Omega)} \leq \varepsilon \tag{9.14}$$

by applying (9.12) with $v(\cdot, \tau)$ replaced by $\tilde{y}(\cdot, \tau)$ and w_1 replaced by $w_1 - \hat{y}(\cdot, \tau)$ (we are neglecting certain technical refinements here; see [37] Théorème 3). Supposing (9.14) to be achieved with a control $\tilde{u}$ with $\|\tilde{u}\|_{L^2(\Gamma_1 \times [0, \tau])} \leq R$, Henry defines a map $\mathcal{T}_\varepsilon(z) = \{y = y(x, t) \mid y$ satisfies (9.13) and boundary conditions of the form (9.11) with $\|u\|_{L^2(\Gamma_1 \times [0, \tau])} \leq R$ and $y(\cdot, \tau)$ satisfies (9.14)$\}$. Actually $\mathcal{T}_\varepsilon$ is a multi-valued map because we do not know that u is unique. For the purposes of this expository article we may think of $\mathcal{T}_\varepsilon : L^2([0, \tau], L^2_\rho(\Omega)) \to L^2([0, \tau]; L^2_\rho(\Omega))$ but Henry uses a more complicated space, namely $H^{3/2, 3/4}(\Omega \times [0, \tau])$.

A fixed point of the multivalued map $\mathcal{T}_\varepsilon$ is a point y such that $y \in \mathcal{T}_\varepsilon(y)$. It is clear that such a fixed point is a solution of the nonlinear system (9.10), (9.11) and satisfies $\|y(\cdot, \tau) - w_1\|_{L^2_\rho(\Omega)} \leq \varepsilon$. In [37] this fixed point is shown to exist by establishing that $\mathcal{T}_\varepsilon$ is upper semi-continuous in an appropriate sense, after which application of the Kakutani-Tychonoff theorem [5] gives the desired result.

The whole field of nonlinear controllability is very much in its infancy and promises to provide grist for many a mathematical mill for some time to come.

9(b) Linear-Quadratic Regulator Theory

We have noted in Section 1 the great impact which the linear-quadratic optimal control theory originating with Kalman and Bucy [47], [48] has had on modern control systems design. Because the theory is heavily dependent on stabilizability and hence, in most cases, on controllability, it is appropriate that we should give some attention to it here. Our treatment is necessarily brief and incomplete.

We will not discuss the theory as it applies to parabolic equations because that area has been so fully and so successfully covered by J. L. Lions in his landmark book [57]: paraphrasing Shakespeare we might say that we "honor his contributions in exclusion" from our less complete discussion. Lions has treated hyperbolic problems as well but the discussion in that case is necessarily less complete because the required controllability results were not available when the book was written.

The most complete controllability results for hyperbolic equations are those for the linear symmetric hyperbolic system in two independent variables discussed in Sections 3 and 4. It is for that system that we shall pose and discuss the linear-quadratic problem here. The treatment follows the general lines presented in [91].

We consider then the system

$$\frac{\partial w}{\partial t} = A(x)\frac{\partial w}{\partial x} + B(x)w \quad , \tag{9.15}$$

$$w^-(0,t) = D_0\, w^+(0,t) \quad , \tag{9.16}$$

$$w^+(1,t) = D_1\, w^-(1,t) + Du(t) \quad , \tag{9.17}$$

all matrices appearing here and all notational conventions to be used hereafter being those of Section 3 with the same hypotheses as set forth there.

Let $T > 0$. Corresponding to an initial state

$$w(\cdot, 0) = w_0 \in L^2([0,1]; E^n) \tag{9.18}$$

and control $u \in L^2[0,\infty)$; we define a quadratic cost functional

$$J(u, w_0, T) = \int_0^T \Big[\int_0^1 \int_0^1 (w(x,t),\ W(x,\xi)\, w(\xi,t))_{E^n}\, dx d\xi$$

$$+ (u(t),\ Uu(t))_{E^q}\Big] dt \quad , \tag{9.19}$$

wherein we assume the $q \times q$ matrix U is symmetric and positive definite while the continuous $n \times n$ matrix function $W(x,\xi)$ satisfies (* denoting adjoint)

$$W(x,\xi) = W(\xi,x)^*, \quad 0 \le x \le 1, \quad 0 \le \xi \le 1$$

and

$$\int_0^1 \int_0^1 (w(x),\ W(x,\xi)\, w(\xi))_{E^n}\, dx d\xi \ge 0, \quad w \in L^2([0,1]; E^n) \ .$$

In (9.19) it is, of course, assumed that w and u together satisfy (9.16)-(9.18).

The conditions necessary and sufficient in order that a pair $\hat{w}$, $\hat{u}$ should (uniquely) minimize (9.19) are as follows. Let $v = v(x,t)$ solve

the adjoint equation (cf. (3.23), modified with a nonhomogeneous term)

$$\frac{\partial v}{\partial t} = A(x)\frac{\partial v}{\partial x} - (B^*(x) - A'(x)v - \int_0^1 W(x,\xi)\,\hat{w}(\xi,t)d\xi \qquad (9.20)$$

with boundary conditions of the form (3.24), (3.25) and

$$v(\cdot, T) = 0 \quad . \qquad (9.21)$$

Then (cf. (3.4), (3.5) for +, - notation)

$$\hat{u}(t) = -U^{-1} D^T A^+(1)\, v^+(1,t), \qquad t \in [0,T] \quad . \qquad (9.22)$$

The equations (9.15), (9.20) with initial and terminal conditions (9.18), (9.21), respectively, together with the noted boundary conditions, are coupled by (9.22) and hence constitute a non-trivial two point boundary value problem for the partial differential equations involved. This is a very complicated mathematical entity; if we were forced to work within this framework the results which we could establish would be severely limited. Fortunately we have the earlier works cited above which provide the critical notions needed to make further progress. The most important idea, which comes from Kalman's solution [47] of the Hamilton-Jacobi equation associated with this optimization problem is the possibility of expressing the adjoint solution v as a linear function of the optimal system state $\hat{w}$:

$$v(x,t) = \int_0^1 Q(x,\xi,t)\,\hat{w}(\xi,t)d\xi \quad . \qquad (9.23)$$

Substituting (9.23) into (9.20) and using (9.15) one obtains for the matrix Q the partial differential equation

$$\frac{\partial Q}{\partial t} = A(x) \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial \xi} A(\xi) + P(Q) \tag{9.24}$$

where

$$\begin{aligned}(P(Q(\cdot, \cdot, t)))(x, \xi) &= -(B(x)^{*} - A'(x))\, Q(x, \xi, t) \\ &- Q(x, \xi, t)(B(\xi)) - W(x, \xi) \\ &+ Q_{+}(x, 1, t)\, A^{+}(1)\, DU^{-1}\, D^{T} A^{+}(1)\, Q^{+}(1, \xi, t) \ .\end{aligned} \tag{9.25}$$

The terminal condition is

$$Q(x, \xi, T) \equiv 0 \tag{9.26}$$

and Q satisfies boundary conditions (cf. (3.24), (3.25))

$$Q^{+}(0, \xi, t) = -(A^{+}(0))^{-1}\, D_0^T\, A^{-}(0)\, Q^{-}(0, \xi, t) \ , \tag{9.27}$$

$$Q_{+}(x, 0, t) = -Q_{-}(x, 0, t)\, A^{-}(0)\, D_0 (A^{+}(0))^{-1} \ , \tag{9.28}$$

$$Q^{-}(1, \xi, t) = -(A^{-}(1))^{-1}\, D_1^T\, A^{+}(1)\, Q^{+}(1, \xi, t) \ , \tag{9.29}$$

$$Q_{-}(x, 1, t) = -Q_{+}(x, 1, t)\, A^{+}(1)\, D_1 (A^{-}(1))^{-1} \ . \tag{9.30}$$

Assuming the system (9.24) - (9.30) can be solved for Q, the formula (9.22) relating the optimal control to the "adjoint" solution v becomes a linear feedback control law

$$\hat{u}(t) = -U^{-1}\, D^T\, A^{+}(1) \int_0^1 Q^{+}(1, \xi, t)\, \hat{w}(\xi, t) d\xi \ . \tag{9.31}$$

The system (9.24)-(9.30) has a very complicated appearance, (9.24), (9.25) are nonlinear with the nonlinearities involving boundary values of the solution. Moreover, (9.24) is a linear hyperbolic system in three independent variables – ordinarily far more complicated than

hyperbolic systems in two variables. It turns out, however, that the fact that $A(x)$, $A(\xi)$ are simultaneously diagonal matrices permits definition and use of characteristic curves (as opposed to characteristic surfaces) in much the same way as one does for two independent variables. Moreover, despite its forbidding appearance, (9.24) is semilinear ([11]). As a consequence, the local existence and uniqueness theory for (9.24)-(9.30) for t near T is very standard, requiring no tools more complicated than those already existing in Chap. V of [11].

Complete solution of the optimization problem requires that the solution $Q(x, \xi, t)$ be continued over the whole interval $0 \leq t \leq T$. For this an a priori bound is required and it is provided through the observation that if the problem of minimizing (9.19) is posed on any interval $[\tau, T]$, $0 \leq \tau \leq T$, with initial state

$$w(\cdot, \tau) = w_\tau \in L^2([0,1]; E^n) ,$$

then the minimal cost turns out to be

$$\int_0^1 \int_0^1 (w_\tau(x),\ Q(x, \xi, \tau)\ w_\tau(\xi))_{E^n} \, dx\, d\xi \quad .$$

This cost, by virtue of its minimality, is less than or equal to that realized with use of a fixed linear feedback relation

$$u_K(t) \equiv \int_0^1 K(x)\ w(x, t)dx \tag{9.32}$$

for which the cost is seen to be

$$\int_0^1 \int_0^1 (w_\tau(x),\ Q_K(x, \xi, \tau)\ w_\tau(\xi))_{E^n}\ dx\, d\xi \quad ,$$

wherein Q_K satisfies a system of the form (9.24)-(9.30) but with $P(Q)$ replaced by $P_K(Q_K)$:

$$(P_K(Q_K(\cdot, \cdot, t)))\ (x, \xi) = -W(x, \xi) - K(x)^* \ UK(\xi)$$

$$-K(x)^* D^T A^+(1)\ Q_K(1, \xi, t) - Q_K(x, 1, t)\ A^+(1)\ D\ K(\xi) \quad .$$

The system satisfied by Q_K is thus linear and solutions may be obtained on arbitrary t-intervals. The a priori bound for Q is then

$$\int_0^1 \int_0^1 (w_\tau(x),\ (Q_K(x, \xi, \tau) - Q(x, \xi, \tau))\ w_\tau(\xi))dx\, d\xi \geq 0 \qquad (9.33)$$

and, properly exploited (see [91]), this enables Q to be extended to $[0, T]$.

Of primary interest in this analysis is what happens as $T \to \infty$. If one can show that there is some $\tilde{Q}(x, \xi)$ such that

$$\int_0^1 \int_0^1 (w_\tau(x),\ (\tilde{Q}(x, \xi) - Q_K(x, \xi, \tau))\, w_\tau(\xi))_{E^n} dx\, d\xi \geq 0 \qquad (9.34)$$

for all $w_\tau \in L^2([0,1]; E^n)$, and for $\tau \in [0, T]$ with T arbitrarily large, then via (9.33) we obtain a global bound for Q on arbitrary intervals as well. Combined with the fact that Q is monotone; if $\tau_1 < \tau_2$

$$\int_0^1 \int_0^1 (w(x),\ (Q(x, \xi, \tau_1) - Q(x, \xi, \tau_2))\ w(\xi))_{E^n}\ dx d\xi \geq 0 \quad ,$$

and the fact that the system (9.24)-(9.30) is autonomous, such a bound

allows one to see that as $T \to \infty$ one obtains a constant form linear feedback control law

$$\hat{u}(t) = -U^{-1} D^T A^+(1) \int_0^1 Q_\infty^+(1, \xi)\, \hat{w}(\xi, t)\, d\xi \tag{9.35}$$

which synthesizes the optimal control for the problem with cost functional (cf. (9.19))

$$J(u, w_0) = \int_0^\infty \Big[\int_0^1 \int_0^1 (w(x,t),\ W(x,\xi)\, w(\xi,t))_{E^n}\, dx\, d\xi$$

$$+ (u(t),\ U,\ u(t))_{E^q}\Big] dt\ . \tag{9.36}$$

The matrix Q_∞ satisfies (cf. (9.24)) the time independent equation

$$A(x) \frac{\partial Q_\infty}{\partial x} + \frac{\partial Q_\infty}{\partial \xi} A(\xi) + P(Q) = 0$$

with boundary conditions again of the form (9.27)-(9.30).

Now in order to establish the vital uniform inequality (9.34) one must establish that solutions of (9.15), (9.16), (9.17) with $u = u_K$ determined by (9.32), for some appropriate K, decay appropriately fast as $t \to \infty$. In general this part of the theory remains incomplete. The only case where it can be said to be completely solved is for the system (4.21), (4.22), (4.23) of Section 4 – and even there the situation is complex enough.

Suppose the boundary conditions (4.22), (4.23) are such (see Assumption 3.8 of Section 3) so that α in (4.9) has negative real part. Then the eigenvalues σ_k of the operator (4.8) asymptotically lie on

the line $\mathrm{Re}(z) = \mathrm{Re}(\alpha) < 0$. Feedback of the form (4.50), which agrees with (9.32), may then be used to ensure uniform exponential decay of solutions of the closed loop system. Once we have this we assure the cost (9.19) with $T = \infty$ is finite for each $w_0 \in L^2([0,1]; E^n)$ when the control determined by (4.50), (9.32) with $k = K$ chosen to ensure uniform exponential decay of solution, is used. $J(u_K, w_0)$ takes the form

$$J(u_K, w_0) = \int_0^1 \int_0^1 (w_0(x), \tilde{Q}(x,\xi)\, w_0(\xi))_{E^n}\, dx d\xi$$

with

$$\tilde{Q}(x,\xi) = \lim_{T\to\infty} Q_K(x,\xi,0) \; (= Q_{K,T}(x,\xi,0)) \; .$$

Thus our ability to achieve uniform exponential decay of solutions via linear feedback is critical in obtaining (9.34) which, in turn leads to the possibility of obtaining the control $\hat{u}$ which minimizes (9.19) with $T = \infty$ in the fixed-form linear feedback representation (9.35). We have seen in Section 4 that this ability to determine system behavior via feedback is dependent on our having fully analyzed the controllability of the system beforehand.

It is also possible to carry out this program for the general system (9.15), (9.16), (9.17) if Assumption 3.8 is valid (rather than just (3.64) and (3.65) of Assumption 3.8 which assure $\mathrm{Re}(\alpha) < 0$ in the case of (4.1), (4.2)). But when Assumption 3.8 is satisfied the system (9.15), (9.16), (9.17) is asymptotically stable with $u(t) \equiv 0$ and the situation is much

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less interesting. What we need are results comparable to those of Section 4 for the general system (9.15), (9.16), (9.17). This remains an open problem at present.

If we begin with a system (4.1), (4.2), wherein $Re(\alpha) \geq 0$ (or, more generally, a system (9.15), (9.16), (9.17) wherein the conditions (3.64), (3.65) are not satisfied) we cannot expect to achieve uniform exponential decay of solutions of the closed loop system using distributed feedback, (4.50) or (9.32). In this situation the relevant theory requires modification of the cost functional (9.19) so that it takes the form

$$J(u, w, T) = \int_0^T \int_0^1 (w(x,t),\ W(x)\, w(x,t))_{E^n}\, dx$$

$$+ (u(t),\ U\, u(t))_{E^q}]dt\ . \tag{9.37}$$

Effectively, the weighting matrix function $W(x, \xi)$, previously defined in the square $0 \leq x \leq 1$, $0 \leq \xi \leq 1$, is replaced by a distribution with support on the diagonal line segment $x = \xi$, $0 \leq x \leq 1$. The solution Q of the Riccati partial differential system corresponding to (9.24)-(9.30) is then distribution valued also and the entire system is immensely complicated. The feedback relations considered take the form

$$u(t) = \hat{K}\, w^-(1,t) + \int_0^1 K(x)\, w(x,t)dx$$

and, in the feedback relation (cf. (9.31)) characterizing the optimal control for (9.37) $\hat{K}$ and K are expressed in terms of Q by very

complicated formulae. The theory here is quite incomplete, particularly with regard to the regularity of the solution Q. The most complete extant treatment is given by H. L. Koh in his thesis [50]. He more or less completely analyzes systems of the form (4.1), (4.2) such as we have treated in Section 4 and goes on to give less complete, but still significant, results for more general systems. The complexity of the calculations are daunting but the basic scheme outlined above for the simpler problem is adhered to and controllability, together with related stabilizability (particularly in connection with Theorem 3.9) continues to play the decisive role.

9(c) Time Optimal Control

We begin with a little background in the context of the finite dimensional system

$$\dot{x} = Ax + Bu, \quad x \in E^n, \quad u \in U, \tag{9.38}$$

where we shall suppose that U is a bounded polyhedron in E^m. Two points $x_0, x_1 \in E^n$ are given and one seeks, if possible, to steer from x_0 to x_1, with a control $u \in L^2[0,T]$ whose values lie in U for each $u \in [0,T]$, and, most importantly, with T as small as possible. A control $\hat{u}$ realizing this objective is a time optimal control. Assuming there is any finite interval $[0,\tilde{T}]$ and control $\tilde{u} \in L^2[0,\tilde{T}]$ with values in U, one can establish (see [76], [77], [55]) the existence of the optimal time T and the existence of an optimal control $\hat{u}$. Moreover,

$\hat{u}$ is characterized by the fact that for almost all $t \in [0, T]$ $\hat{u}(t)$ solves the linear programming problem

$$\min_{u \in U} (\psi(t), Bu)_{E^n} \tag{9.39}$$

where ψ is a non-trivial solution of the adjoint equation

$$\dot{\psi} = -A^* \psi .$$

With appropriate "normality" assumptions, the fact that $\hat{u}$ solves (9.39) guarantees that we have

$$[0, T] = \bigcup_{i=1}^{N} I_i, \quad \hat{u}(t) = v_i, \quad t \in \mathrm{Int}(I_i) ,$$

where the I_i are closed intervals, overlapping at most at common boundary points, and the v_i are vertices of the polyhedron U. This result is known as the "bang-bang" principle [51] since it implies that the time optimal control $\hat{u}$ "bangs around" from one vertex of U to another during the control period. The simplicity of this type of control action led to great interest on the part of control engineers during (roughly) the period 1955-1965 but that interest waned somewhat as it was realized that synthesis of $\hat{u}(t)$, i.e., expression in terms of the system state in a (non-linear) "feedback" form, was almost hopelessly complicated for all but the simplest systems and that there were many practical drawbacks associated with such abrupt control switching. Nevertheless the theory has found significant application in a number of areas and, in addition, has appealing mathematical simplicity and

elegance. It is natural that a number of attempts have been made to extend the theory to distributed systems.

By the time the reader has reached this point in the paper he will not be surprized to find that time optimal control theory is very different for different classes of partial differential equations. For hyperbolic systems there is, strictly speaking, no true "bang-bang" principle at all. This is easy to see in the context of the system studied in Section 4. With initial and terminal states given at $t = 0$ and $t = 2$, respectively, there is exactly one control, given by formula (4.18), i.e.,

$$u(t) = \sum_{\ell=-\infty}^{\infty} \left(\frac{c_\ell}{g_\ell}\right) \overline{p_\ell(t)} \; .$$

Now it can be shown that the p_ℓ are bounded functions. Hence if the coefficients c_ℓ, determined by the initial and terminal states, decrease rapidly enough, the control $u(t)$ will satisfy any given à priori bound, e.g.,

$$-1 \leq u(t) \leq 1 \quad (\text{i.e., cf. } (9.38), \quad U = [-1,1]) \; . \tag{9.40}$$

In general the time $t = 2$ is optimal and $u(t)$, given by (4.18) is the time optimal control because it is the unique control in $L^2[0,T]$. But it is clear that $u(t)$ is not ordinarily a function assuming only the values ± 1; indeed, if the c_ℓ decrease rapidly enough, $u(t)$ will possess any desired degree of differentiability in $[0,2]$. Thus no all-inclusive bang-bang principle of the type which we have for finite dimensional systems (9.38) obtains.

We present here a bang-bang time optimal control result for the parabolic control system of Section 7, i.e.,

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x} + r(x)w = g(x)\, u(t) \quad , \tag{9.41}$$

$$a_0\, w(0,t) + b_0 \frac{\partial w}{\partial x}(0,t) = 0, \quad a_1 w(1,t) + b_1 \frac{\partial w}{\partial x}(1,t) = 0 \quad . \tag{9.42}$$

The result also applies, without essential modification, to the boundary control system described in that section.

We consider, then, the system (9.41), (9.42) with the added restriction

$$-1 \leq u(t) \leq 1 \quad . \tag{9.43}$$

For convenience (and without loss of generality) we assume here that the initial state is

$$w(\cdot, 0) = 0 \tag{9.44}$$

and our goal is to reach, in minimal time τ, a given terminal state

$$w(\cdot, \tau) = w_1 \quad .$$

Disregarding (9.43) for the moment, this can be done, e.g., if w_1 is a state in the domain of the operator $\exp(A_1 (-L)^{\frac{1}{2}})$ as described in Theorem 7.3; and it is a genuine curiosity of the theory that it is only for those states, which one has constructively shown to be reachable, that the bang-bang control principle can be established. Our approach is a simpler version of a more general result due to Fattorini [26] and follows along the same lines as an earlier result of Egorov [18], who

also confronted this restriction on the result. Whether or not the result holds for all reachable states appears to be completely unknown at this writing.

The basic idea is to impose a particular topology upon the space of reachable states which we have identified in Theorem 7.3. To do this we again resort to the method of biorthogonal functions. In Section 7 we noted the existence of biorthogonal functions with (cf. (7.33))

$$\|p_k\|_{L^2[0,\tau]} \leq K_0 \exp(K_1 \omega_k) \; .$$

It turns out that the p_k can be modified slightly (see [27]) to another set of biorthogonal functions q_k for the $e^{-\lambda_k t}$ such that each $q_k \in C[0,\tau]$

$$\|q_k\|_{C[0,\tau]} = \max_{t\in[0,\tau]} |\tilde{q}_k(t)| \leq \tilde{K}_0 \exp(K_1 \omega_k) \qquad (9.45)$$

for some $\tilde{K}_0 > 0$. Or, the same result can be achieved <u>ab ovo</u> via the Fourier transform method (see [27] again) which we employed in a slightly modified setting in Section 8. Let us put

$$b_k = \|q_k\|_{C[0,\tau]} \; . \qquad (9.46)$$

We have noted that elements $w \in L^2[0,1]$ can be expanded in series

$$w = \sum_{k=1}^{\infty} \mu_k \phi_k \qquad (9.47)$$

where the ϕ_k are the eigenfunctions of the Sturm-Liouville operator $Lw = -\dfrac{\partial^2 w}{\partial x^2} + r(x)w$. We likewise have, for the element g in (9.41),

$$g = \sum_{k=1}^{\infty} g_k \phi_k \ .$$

Assuming that no $g_k = 0$, we define

$$\|w\|_B = \sum_{k=1}^{\infty} \frac{b_k}{|g_k|} |\mu_k| \tag{9.48}$$

for those w in the subspace $B \subseteq L^2[0,1]$ which thereby yield a finite sum. With the norm (9.48) B becomes a Banach space. Each $w_1 \in B$ lies in the set of reachable states for, using now the q_k instead of the p_k, w_1 is reached by the control

$$u(t) = \sum_{k=1}^{\infty} \frac{\mu_k}{g_k} q_k(\tau - t)$$

and for all $t \in [0, \tau]$

$$|u(t)| \leq \sum_{k=1}^{\infty} \left| \frac{\mu_k}{g_k} \right| b_k = \|w_1\|_B \ . \tag{9.49}$$

Finally we note that one can establish, much along the lines of Fattorini's proof of time invariance of the reachable set, that the above definition of B is actually independent of the time τ.

We are now in a position to state and prove the main result.

Theorem 9.1. Suppose a state $w_1 \in B$ is reached at time $\tau > 0$ from (9.44) via (9.41), (9.42) with a measurable control $\hat{u}$ which satisfies (9.43) and suppose this time τ is minimal with respect to all measurable u satisfying (9.43). (We remark that while there is no minimal time τ

for square integrable controls it can be shown to exist for controls satisfying (9.43). See, e.g. [57].) Then $|\hat{u}(t)| = 1$ for almost all $t \in [0, \tau]$. Moreover, the time optimal control $\hat{u}(t)$ is characterized by the formula

$$\hat{u}(t) = \operatorname{sgn} \eta(t), \quad \eta(t) \neq 0$$

where $\eta(t)$ is real analytic for $t < \tau$ with discrete zeros accumulating only at τ.

The proof of Theorem 9.1 involves three lemmas.

Lemma 9.2. For any $T > 0$ let $R(T)$ denote the subset of B consisting of states $\mathbf{w}$ reachable from (9.44) via (9.41), (9.42) during $[0, T]$ with use of a measurable control u satisfying (9.43). Then $R(T)$ contains a neighborhood of the origin in B.

Proof. Immediate, for $R(T)$ includes all $w \in B$ with $\|w\|_B \leq 1$ by virtue of (9.48).

Lemma 9.3. If $w_1 \in R(\tau)$ but not to any $R(T)$ for $T < \tau$ (so that τ is, indeed, the minimal time to reach w_1) then w_1 belongs to the boundary of $R(\tau)$ in the Banach space B.

Proof. If not, then $w_1 \in \operatorname{Int}(R(\tau))$ in B and $R(\tau)$ includes the set

$$N_\varepsilon(w_1) = \{w_1 + w \mid \|w\|_B < \varepsilon\}$$

for some $\varepsilon > 0$. Then there is some $r < 1$ such that $\frac{1}{r} w_1 \in R(\tau)$ and hence $\frac{1}{r} w_1$ is reachable in time τ. Assuming the expansion (9.47)

for w_1, $\frac{1}{r} w_1 \in R(\tau)$ means there is some u satisfying (9.44) on $[0, \tau]$ such that

$$g_k \int_0^{\tau} e^{-\lambda_k(\tau - t)} u(t)dt = \frac{1}{r} \mu_k, \qquad k = 1, 2, 3, \ldots .$$

Now let $\delta > 0$ and define

$$w_\delta = \sum_{k=1}^{\infty} \nu_k \varphi_k ,$$

$$\nu_k = g_k \int_0^{\tau-\delta} e^{-\lambda_k(\tau-\delta-t)} u(t+\delta)dt .$$

It is clear that $w_\delta \in R(\tau-\delta)$ if it belongs to B. To show the latter we compute

$$\sum_{k=1}^{\infty} \frac{b_k}{|g_k|} \left| \frac{1}{r} \mu_k - \nu_k \right|$$

$$= \sum_{k=1}^{\infty} \frac{b_k}{|g_k|} \left| g_k \int_0^{\tau} e^{-\lambda_k(\tau-t)} u(t)dt - g_k \int^{\tau-\delta} e^{-\lambda_k(\tau-\delta-t)} u(t+\delta)dt \right|$$

$$= \sum_{k=1}^{\infty} b_k \left| \int_0^{\delta} e^{-\lambda_k(\tau-t)} u(t)dt \right| \le \sum_{k=1}^{\infty} b_k \, \delta \, e^{-\lambda_k(\tau-\delta)} . \tag{9.50}$$

Now (9.45), (9.46) show the last sum to be finite so $w_1 - w_\delta$ lies in B and thus, as we have noted, in $R(\tau - \delta)$. It is also clear, from (9.50) that we can make $\|\frac{1}{r} w_1 - w_\delta\|_B$ as small as we wish by taking $\delta > 0$ small. Then we can also make $\|w_1 - r w_\delta\|_B$ as small as we wish. But, since $r < 1$, $r w_\delta$ clearly lies in $R(\tau - \delta)$ and is reached with the control $ru(t+\delta)$ which satisfies $-1 < -r \le u(t+\delta) \le r < 1$ for $t \in [0, \tau-\delta]$.

Using controls $v(t)$ with $-(1-r) \leq v(t) \leq 1-r$, $t \in [0, \tau-\delta]$, we conclude from Lemma 9.2 that $R(\tau-\delta)$ also includes the set $\{r w_\delta + w \mid \|w\|_B \leq 1-r\}$. Taking δ small enough so that $\|w_1 - r w_\delta\| \leq 1-r$ we have $w_1 \in R(\tau-\delta)$. This is contrary to our assumption that τ is the minimal time for which $w_1 \in R(\tau)$. We conclude w_1 belongs to the boundary of $R(\tau)$ in B and the proof of the lemma is complete.

Lemma 9.4. *There is a continuous linear functional* $\eta \in B^*$ *with the representation*

$$\eta(w) = \eta\left(\sum_{k=1}^{\infty} \theta_k \varphi_k\right) = \sum_{k=1}^{\infty} \eta_k \theta_k, \quad w \in B \tag{9.51}$$

such that

(i) $\quad |\eta_k| \leq K \dfrac{b_k}{|g_k|}, \quad k = 1, 2, 3, \ldots$ for some $K > 0$;

(ii) $\quad \eta(w_1) \geq \eta(w), \quad w \in R(\tau)$.

Proof. The existence of η satisfying (ii) follows from Mazur's separation theorem ([17]) after we take note that $R(\tau)$ is convex and, from Lemma 9.2, includes a neighborhood of the origin.

Now (9.48) clearly shows that

$$\left\| \frac{|g_k|}{b_k} \varphi_k \right\|_B = 1, \quad k = 1, 2, 3, \ldots$$

from which we conclude

$$\eta\left(\frac{|g_k|}{b_k} \varphi_k\right) = \frac{|g_k|}{b_k} \eta_k \leq \|\eta\|_{B^*}$$

and we have (i) with $K = \|\eta\|_{B^*}$.

We can now complete the

Proof of Theorem 9.1. We have

$$w_1 = \sum_{k=1}^{\infty} g_k \left(\int_0^{\tau} e^{-\lambda_k(\tau - t)} \hat{u}(t) dt \right) \varphi_k$$

where $\hat{u}$ is a control steering 0 to w_1 during the minimal time interval $[0, \tau]$. For any other control u satisfying (9.44) on $[0, \tau]$ we have the endpoint

$$w(\tau) = \sum_{k=1}^{\infty} g_k \left(\int_0^{\tau} e^{-\lambda_k(\tau - t)} u(t)\, dt \right) \varphi_k \quad .$$

If $w(\tau) \in R(\tau)$, Lemma 9.4 applies to give

$$\eta(w_1 - w(\tau)) \geq 0 \tag{9.52}$$

for some $\eta \in B^*$. Using (9.51) this becomes

$$\int_0^{\tau} \left(\sum_{k=1}^{\infty} \eta_k g_k e^{-\lambda_k(\tau - t)} \right) (\hat{u}(t) - u(t)) dt \geq 0 \quad ,$$

the change in order of summation and integration being easily justified from (i) of Lemma 9.4. That property of the η_k also shows that

$$\eta(z) = \sum_{k=1}^{\infty} \eta_k g_k e^{-\lambda_k(\tau - z)}$$

is analytic in the open left half plane $\mathrm{Re}(z) < \tau$. Hence its zeros on the positive real axis are discrete with no accumulation point other than $z = 0$.

Consider then any interval (t_1, t_2) with $0 \le t_1 < t_2 < \tau$ wherein $\eta(t)$ is of one sign, w.l.o.g. positive. If we let

$$u(t) = \begin{cases} \hat{u}(t), & t \notin (t_1, t_2) \\ 1, & t \in (t_1, t_2) \end{cases} \tag{9.53}$$

then u satisfies (9.44) on $[0, \tau]$ and, since $t_2 < \tau$, the solution $\hat{w}(t) - w(t)$ of (9.41), (9.42) corresponding to $\hat{u}(t) - u(t)$ satisfies

$$\frac{\partial(\hat{w} - w)}{\partial t} - \frac{\partial^2(\hat{w} - w)}{\partial x^2} + r(x)(\hat{w} - w) = 0$$

for $t_2 \le t \le \tau$. This last fact shows quite readily that $\hat{w}(\tau) - w(\tau) = w_1 - w(\tau)$ lies in B and hence, since we know $w_1 \in B$, that $w(\tau) \in B$ as required for (9.52). But (9.52), with u given by (9.53), becomes

$$\int_{t_1}^{t_2} \eta(t)(\hat{u}(t) - 1)dt \ge 0$$

and, since we have assumed $\eta(t) > 0$ in t_1, t_2, we must have $\hat{u}(t) \equiv 1$, $t \in (t_1, t_2)$. From this we see that for all $t \in [0, \tau)$ with $\eta(t) \ne 0$ we have

$$\hat{u}(t) = \text{sgn}\, \eta(t)$$

and the proof is complete.

We note that $\eta(t)$ has the form $(y(\cdot, t), g)_{L^2[0,1]}$ where

$$\frac{\partial y}{\partial t} + \frac{\partial^2 y}{\partial x^2} - r(x)\, y = 0, \quad x \in [0, 1], \quad t < \tau .$$

In general $y(\cdot, \tau) \notin L^2[0,1]$, however. We have noted that the zeros of $\eta(t)$ are discrete, accumulating, if at all, only at $t = \tau$. Hence the optimal control $\hat{u}(t)$ assumes the values ± 1 on an at most countable sequence $\{I_i\}$ of intervals separated by "switching times" t_i with $\lim_{i \to \infty} t_i = \tau$. Thus $\hat{u}(t)$ is, indeed, a "bang-bang" control.

The role played by controllability in this theory is very clear. It is necessary to define a space of reachable states whose topology is strong enough so that the sets $R(\tau)$ defined above have interior points, allowing Mazur's theorem to be applied, and yet also weak enough so that $R(\tau)$ moves continuously with τ, permitting one to conclude the result of Lemma 9.3. These conflicting requirements on the topology of B require that a great deal be known about the reachable points and their relationship to the controls which achieve them. Similar observations were made by Quinn in [78] in connection with other time optimal control problems for distributed systems.

We note in conclusion that a numerical procedure for determining the switching times t_i described above has been proposed by Goldwyn Sriram and Graham in [32] with impressive numerical success. The theoretical basis for the method appears, however, to require some strengthening.

REFERENCES

[1] Adams, R. A.: Sobolev Spaces, Academic Press, New York, San Francisco, London, 1975.

[2] Agmon, S.: Elliptic Boundary Value Problems, Van Nostrand, Princeton, 1965.

[3] Baras, J. S., R. W. Brockett and P. A. Fuhrmann: State-space models for infinite dimensional systems, IEEE Trans. Automatic Control, AC-19 (1974), 693-700.

[4] Baras, J. S. and R. W. Brockett: H^2-functions and infinite-dimensional realization theory, SIAM J. Control, 13(1975), 221-241.

[5] Browder, F. E.: The fixed point theory of multi-valued mappings in topological vector spaces, Math. Ann., 177 (1968), 283-301.

[6] Brunovsky, P.: A classification of linear controllable systems, Kybernetika, 6(1970), 173-188.

[7] Butkovskiy, A. G.: Theory of Optimal Control of Distributed Parameter Systems, American Elsevier Pub. Co., New York, 1969.

[8] Chewning, W. C.: Controllability of the nonlinear wave equation in several space variables, SIAM J. Control, 14(1976), 19-25.

[9] Cirina, M. A.: Boundary controllability of nonlinear hyperbolic systems, SIAM J. Control, 7(1969), 198-212.

[10] Courant, R. and D. Hilbert: Methods of Mathematical Physics, Vol. I, John Wiley and Sons, Inc., New York, 1953.

[11] Courant, R. and D. Hilbert: Methods of Mathematical Physics, II: Partial Differential Equations, Interscience Pub. Co., New York, 1962.

[12] Datko, R.: Extending a theorem of A. M. Liapounov to Hilbert space, J. Math. Anal. Appl., 32(1970), 610-616.

[13] Dieudonné, J.: Foundations of Modern Analysis, Academic Press, New York, 1960.

[14] Dolecki, S.: Observability for the one-dimensional heat equation, Studia Math., 48(1973), 291-305.

[15] __________: A classification of controllability concepts for infinite-dimensional linear systems, Preprint No. 79, Institute of Mathematics, Polish Academy of Sciences, Oct. 1975.

[16] Dolecki, S. and D. L. Russell: A general theory of observation and control, Tech. Summ. Rept. #1519, Math. Res. Ctr., Univ. of Wisc., Madison, August 1975 (to appear in SIAM J. Control and Optimization.)

[17] Dunford, N. and J. T. Schwartz: Linear Operators, Part I.: General Theory, Interscience Pub. Co., New York, 1958.

[18] Egorov, Yu. V.: Some problems in the theory of optimal control, Soviet Mathematics, 3(1962), 1080-1084.

[19] __________: Ẑ. Vycisl. Mat. Fiz., 5(1963), 887-904.

[20] Fattorini, H. O.: Control in finite time of differential equations in Banach space, Comm. Pure Appl. Math., 19(1966), 17-34.

[21] Fattorini, H. O.: Some remarks on complete controllability, SIAM J. Control, 4(1966), 686-694.

[22] ______________: On complete controllability of linear systems, J. Differential Equations, 3(1967), 391-402.

[23] ______________: Boundary control systems, SIAM J. Control, 6(1968), 349-388.

[24] ______________: Local controllability of a nonlinear wave equation, Math. Systems Theory, 9(1975), 30-45.

[25] ______________: Boundary control of temperature distributions in a parallelopipedon, SIAM J. Control, 13(1975), 1-13.

[26] ______________: The Time-Optimal Problem for Boundary Control of the Heat Equation, in Calculus of Variations and Control Theory, Academic Press, New York, San Francisco, London, 1976.

[27] ______________ and D. L. Russell: Exact controllability theorems for linear parabolic equations in one space dimension, Arch. Rational Mech. Anal., 43(1971), 272-292.

[28] ______________________________: Uniform bounds on biorthogonal functions for real exponentials with an application to the control theory of parabolic equations, Quart. Appl. Math., 32(1974), 45-69.

[29] Friedman, A.: Partial Differential Equations, Holt, Rinehart and Winston, Inc., New York, 1969.

[30] Gal'chuk, L. I.: Optimal control of systems described by parabolic equations, SIAM J. Control, 7(1969), 546-558.

[31] Garabredian, P.: Partial Differential Equations, John Wiley and Sons, Inc., New York, 1964.

[32] Goldwyn, R. M., K. P. Sriram and M. Graham: The optimal control of a linear diffusion process, SIAM J. Control, 5(1967), 295-308.

[33] Graham, K. D. and D. L. Russell: Boundary value control of the wave equation in a spherical region, SIAM J. Control, 13(1975), 174-196.

[34] Hale, J. K.: Dynamical systems and stability, J. Math. Anal. Appl., 26(1969), 39-59.

[35] Helton, J. W.: Discrete time systems, operator models, and scattering theory, J. Functional Analysis, 16(1974), 15-37.

[36] ____________: Systems with infinite dimensional state space: the Hilbert space approach, Proc. IEEE, 64(1976), 145-160.

[37] Henry, J.: Étude de la controllabilité de certaines equations paraboliques non-lineaires, to appear in Proc. IFIP Working Conf. on Distributed Parameter Systems, Rome, June 1976.

[38] Hille, E.: Analytic Function Theory, Vol. I, Ginn and Co., Boston, 1959.

[39] Hille, E. and R. S. Phillips: Functional Analysis and Semi-groups, Amer. Math. Soc. Colloq. Publ., Vol. 31, Providence, R. I., 1957.

[40] Holmgren, E.: Über systeme von linearen partiellen differentialgleichungen, Ofversigt af Kongl. Vetenskapsakad. Förk, 58(1901), 91-103.

[41] Hörmander, L.: Linear Partial Differential Operators, Springer-Verlag, Heidelberg-Berlin, 1963.

[42] Horowitz, I. and U. Shaked: Superiority of transfer function over state-variable methods in linear time-invariant feedback system design, IEEE Trans. Automatic Control, AC-20 (1975), 84-97.

[43] John, F.: On linear partial differential equations with analytic coefficients - unique continuation of data, Comm. Pure Appl. Math., 2(1949), 209-253.

[44] Johnsson, L.: Distributed parameter systems: an annotated bibliograph — to April 1971, Dept. of Eng., Univ. of Cal. L. A., Int. Sci. Rept., Aug. 1971.

[45] Kaczmarz, S. and H. Steinhaus: Theorie der Orthogonalreihen, Monografje Matematyczne, Tom VI, Warsaw-Lwow 1935.

[46] Kadec, M. I.: The exact value of the Paley-Wiener constant, Soviet Mathematics, 5(1964), 559-561.

[47] Kalman, R. E.: Contributions to the theory of optimal control, Bol. Soc. Mat. Mexicana, 5(1960), 102-119.

[48] ________ and R. S. Bucy: New results in linear prediction and filtering theory, J. Basic Eng., Trans. ASME Ser. D, 83(1961), 95-100.

[49] Kleinman, D. L., An easy way to stabilize a linear constant system, IEEE Trans. Automatic Control, AC-15(1970), 692.

[50] Koh, H. L.: Structure of Riccati equation solutions in optimal boundary control of hyperbolic equations with quadratic cost functionals, Tech. Summ. Rept. No. 1642, Math. Res. Ctr., Univ. of Wisc., Madison, June 1976.

[51] La Salle, J. P.: The 'bang-bang' principle, Proc. IFAC Congress, Moscow, 1960, Butterworths and Co., Ltd., London, 1961, Vol. 1, 493-497.

[52] __________ and S. Lefschetz: Stability by Liapounov's Direct Method with Applications, Academic Press, New York, London, 1961.

[53] Lax, P. D., C. S. Morawetz and R. S. Phillips: Exponential decay of solutions of the wave equation in the exterior of a star-shaped obstacle, Comm. Pure Appl. Math., 16(1963), 477-486.

[54] __________ and R. S. Phillips: Scattering Theory, Academic Press, New York, London, 1967.

[55] Lee, E. B. and L. W. Markus: Foundations of Optimal Control Theory, John Wiley and Sons, Inc., New York, 1967.

[56] Levinson, N.: Gap and density theorems, Amer. Math. Soc. Colloq. Publ., 26, Providence, R. I., 1940

[57] Lions, J. L.: Optimal Control of Systems Governed by Partial Differential Equations, translated by S. K. Mitter, Springer-Verlag, New York-Heidelberg-Berlin, 1971.

[58] __________ and E. Magenes: Problèmes aux limites nonhomogènes et applications, Vol. 1, Dunod, Paris, 1968.

[59] Lions, J. L. and E. Magenes: Problèmes aux limites nonhomogènes et applications, Vol. 2, Dunod, Paris, 1968.

[60] Lukes, D. L.: Stabilizability and optimal control, Funkcial.Ekvac., 11(1968), 39-50.

[61] __________: Global controllability of nonlinear systems, SIAM J. Control, 10(1972), 112-126.

[62] Luxemburg, W. A. and J. Korevaar: Entire functions and Muntz-Szász type approximation, Trans. Amer. Math. Soc., 57(1971), 23-37.

[63] Mac Camy, R. C., V. J. Mizel and T. I. Seidman: Approximate boundary controllability of the heat equation, J. Math. Anal. Appl., 23(1968), 699-703.

[64] ____________________: II, Ibid., 28(1969), 482-492.

[65] Markus, L.: Controllability for nonlinear processes, SIAM J. Control, 3(1965), 78-90.

[66] Mihailov, V. P.: Riesz bases in $L_2[0,1]$, Soviet Mathematics, 3(1962), 851-855.

[67] Mizel, V. J. and T. I. Seidman: Observation and prediction for the heat equation, J. Math. Anal. Appl., 28(1969), 303-312.

[68] _______________: II, 38(1972), 149-166.

[69] Mizohata, S.: Unicite du prolongement des solutions pour quelques opérateurs différentiels paraboliques, Mém. Coll. Sci. Univ. Kyoto Série A, 31(1958), 219-239.

[70] Morawetz, C. S.: The decay of solutions of the exterior initial-boundary value problem for the wave equation, Comm. Pure Appl. Math., 14(1961), 561-568.

[71] _______________: Exponential decay of solutions of the wave equation, Comm. Pure Appl. Math., 19(1966), 439-444.

[72] _______________ : Decay for solutions of the exterior problem for the wave equation, Comm. Pure Appl. Math., 28(1975), 229-264.

[73] Paley, R. E. A. C. and N. Wiener: Fourier Transforms in the Complex Domain, Amer. Math. Soc. Colloq. Publ., 19, Providence, R. I., 1934.

[74] Pallu de la Barriére, R.: Optimal Control Theory, W. B. Saunders Co., Philadelphia, 1967.

[75] Phillips, R. S.: Dissipative hyperbolic systems, Trans. Amer. Math. Soc., 86(1957), 109-173.

[76] Pontryagin, L. S.: Optimal Control Processes, Amer. Math. Soc. Transl., 18(1961), 321-339.

[77] _______________, Boltyanskii, V. G., Gamkrelidze, R. V., Mischenko, E. F.: The Mathematical Theory of Optimal Processes, Interscience, New York, 1962.

[78] Quinn, J. P.: Time optimal control of linear distributed parameter systems, Thesis, Dept. of Math., University of Wisconsin, August, 1969.

[79] ____________ and D. L. Russell: Asymptotic stability and energy decay rates for solutions of hyperbolic equations with boundary damping, to appear in Proc. Royal Soc. Edinburgh, Series A.

[80] Ralston, J. V., Solutions of the wave equation with localized energy, Comm. Pure Appl. Math., 22(1969), 807-823.

[81] ____________ : Local decay of solutions of conservative first order hyperbolic systems in odd dimensional space, Trans. Amer. Math. Soc., 194(1974), 27-51.

[82] Rauch, J. and M. Taylor: Exponential decay of solutions to hyperbolic equations in bounded domains, Indiana Univ. Math. J., 24(1974), 79-86.

[83] Redheffer, R. M.: Elementary remarks on completeness, Duke Math. J., 35(1968), 103-116.

[84] Riesz, F. and B. Sz.-Nagy: Functional Analysis, Frederick Ungar Pub. Co., New York, 1955.

[85] Robinson, A. C.: A survey of optimal control of distributed-parameter systems, ARL, Wright Patterson AFB, Ohio, Report ARL-0177.

[86] Russell, D. L.: On boundary-value controllability of linear symmetric hyperbolic systems, in Mathematical Theory of Control, Academic Press, New York, 1967, 312-321.

[87] Russell, D. L.: Nonharmonic Fourier series in the control theory of distributed parameter systems, J. Math. Anal. Appl., 18(1967), 542-559.

[88] ____________ : Boundary value control theory of the higher dimensional wave equation, SIAM J. Control, 9(1971), 29-42.

[89] ____________ : Part II, Ibid., 401-419.

[90] ____________: Control theory of hyperbolic equations related to certain questions in harmonic analysis and spectral theory, J. Math. Anal. Appl., 40(1972), 336-368.

[91] ____________: Quadratic performance criteria in boundary control of linear symmetric hyperbolic systems, SIAM J. Control, 11(1973), 475-509.

[92] ____________: A unified boundary controllability theory for hyperbolic and parabolic partial differential equations, Studies in Applied Math., 52(1973), 189-211.

[93] ____________: Exact boundary value controllability theorems for wave and heat processes in star-complemented regions, in Differential Games and Control Theory, Roxin, Liu, Sternberg, Eds. Marcel Dekker, Inc., New York, 1974.

[94] ____________: Decay rates for weakly damped systems in Hilbert space obtained with control-theoretic methods, J. Differential Equations, 19(1975), 344-370.

[95] Russell, D. L.: Canonical forms and spectral determination for a class of hyperbolic distributed parameter control systems, Tech. Summ. Rept. No. 1614, Math. Res. Ctr., Univ. of Wisc., Madison, Feb. 1976. To appear in J. Math. Anal. Appl.

[96] ____________: Differential-delay equations as canonical forms for controlled hyperbolic systems with applications to spectral assignment, to appear in Proc. Conf. Contr. Theory of Distr. Parameter Systems, Naval Surface Weapons Ctr., White Oak, Md., June 1976.

[97] Sakawa, Y.: Controllability for partial differential equations of parabolic type, SIAM J. Control, 12(1974), 389-400.

[98] _________: Observability and related problems for partial differential equations of parabolic type, SIAM J. Control, 13(1975), 14-27.

[99] Schwartz, L.: Étude des sommes d'exponentielles, deuxième édition, Hermann, Paris, 1950.

[100] Seidman, T. I.: Observation and prediction for the heat equation, III, to appear in J. Differential Equations. Announcement: A well-posed problem for the heat equation, Bull. Amer. Math. Soc., 80(1974), 901-902.

[101] ____________: Boundary observation and control for the heat equation, in Calculus of Variations and Control Theory, Academic Press, New York, San Francisco, London, 1976.

[102] Seidman, T. I.: Observation and prediction for the heat equation, IV: Patch observability and controllability, Math. Research Report No. 76-4, Div. Math. and Phys., Univ. Md., Balt. Co., Feb. 1976 (to appear in SIAM J. Control and Optimization).

[103] Slemrod, M.: A note on complete controllability and stabilizability for linear control systems in Hilbert space, SIAM J. Control, 12(1974), 500-508.

[104] ___________: Stabilization of boundary control systems, J. Differential Equations, to appear.

[105] ___________: The linear stabilization problem in Hilbert space, to appear in Arch. Rational Mech. Anal.,

[106] Strauss, W. A.: Dispersal of waves vanishing on the boundary of an exterior domain, Comm. Pure Appl. Math., 28(1975), 265-278.

[107] Triggiani, R.: On the stabilizability problem in Banach space, J. Math. Anal. Appl., 52(1975), 383-403.

[108] ___________: Extensions of rank conditions for controllability and observability to Banach spaces and unbounded operators, SIAM J. Control, 14(1976), 313-338.

[109] Ulrich, D.: "Divided differences and systems of nonharmonic Fourier series", to appear.

[110] Wiener, N.: Extrapolation, Interpolation and Smoothing of Stationary Time Series, M.I.T. Press, Cambridge, 1957.

[111] Wonham, W. M.: Linear Multivariable Control; A Geometric Approach, Springer-Verlag, Berlin, Heidelberg, New York, 1974.